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TOUCHARD POLYNOMIALS APPROACH FOR SOLVING NON- LINEAR FREDHOLM INTEGRO- DIFFERENTIAL EQUATION OF SECOND TYPE

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Abstract

A numerical approach to solve Non- linear Fredholm integro- differential (NLFID) equation of the 1st order and 2nd type has been proposed. The approach was based upon Touchard polynomials. The non-linear Fredholm integro- differential equation was changed into a system of non- linear algebraic equations and solved using Newton repeating approach. The proposed approach was evaluated by displaying three numerical problems, and the approximate numerical solutions were compared with exact solution and four methods in the literature. MATLAB R2018b was used to perform all calculations and graphs.

Keywords: *Non-linear Fredholm integro- differential equation, Touchard polynomials, Numerical solutions, Numerical method, exact solutions.*

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Introduction

The 2nd type of Fredholm integro- differential equations appear in a range of scientific applications, include signal processing and neural network [1]. The standard form of the (NLFID) equation of the 1st order and 2nd type [1, 2, 3 and 4] defined as follows:

$$\begin{cases} B'(\omega) = C(\omega) + \int_0^1 I(\omega,\varphi)K(B(\varphi))d\varphi, \qquad \omega \in [d,e] \\ \qquad \text{with the initial condition } B(0) = b_0 , \end{cases} \qquad \dots(1.1)$$

and $B'(\omega)$ is derivative of $B(\omega)$, the kernels $I(\omega,\varphi)$, and $C(\omega)$ are known valued functions, and $K\big(B(\omega)\big)$ is a non- linear function of $B(\omega)$ to be calculated. Researchers have utilized or established a number of numerical methods to get numerical solutions of the (NLFID) equation, some of which are listed below: in [5] used the direct calculation approach. In [6] applied the Haar wavelet method. In [7] used the (ADM) and (HAM) methods. In [8] used the additional parameters method and anew functions. In [9] applied the rationalized Haar wavelet method. Finally [10] used the (RHFs) method.The following is the order of the paper: Touchard polynomials, Touchard's operational matrix, solution of the (NLFID) equation, test examples with graphs, summary of conclusions and recommendations, and references are all included at the end.

2. Touchard Polynomials (TPs)

Jacques Touchard [11, 12], French mathematician was the first one to discover the (TPs), which are binomial polynomial sequences defined on [0, 1] as follows:

$$\psi_\varrho(\omega) = \sum_{\upsilon=0}^{\varrho} \phi(\varrho,\upsilon)\omega^\upsilon = \sum_{\upsilon=1}^{\varrho} \binom{\varrho}{\upsilon}\omega^\upsilon , \quad \binom{\varrho}{\upsilon} = \frac{\varrho!}{\upsilon!\,(\varrho-\upsilon)!} , \qquad \dots(2.1)$$

where ϱ and υ are the degree and index respectively. The first four polynomials are defined as follows:

$$\psi_0(\omega) = 1,$$

$$\psi_1(\omega) = 1 + \omega,$$

$$\psi_2(\omega) = 1 + 2\omega + \omega^2,$$

$$\psi_3(\omega) = 1 + 3\omega + 3\omega^2 + \omega^3$$

2.1. Touchard's Operational Matrix

Assume that the linear combination of Touchard polynomials defines the approximate solutions to Eq. (1.1)

$$B_\varrho(\omega) = \chi_0\psi_0(\omega) + \chi_1\psi_1(\omega) + \cdots + \chi_\varrho\psi_\varrho(\omega) = \sum_{\upsilon=0}^{\varrho} \chi_\upsilon\psi_\upsilon(\omega), \quad \dots (2.2)$$

the functions $\{\psi_\upsilon(\omega)\}_{\upsilon=0}^{\varrho}$ are Touchard's basis of ϱth degree, as defined in Eq. (2.1), where χ_υ $(\upsilon = 0,1,\dots,\varrho)$ indicates the parameters that can be calculated later . As a dot product, written Eq. (2.2).

$$B_\varrho(\omega) = [\psi_0(\omega) \;\; \psi_1(\omega)\dots\psi_\varrho(\omega)] . \begin{bmatrix} \chi_0 \\ \chi_1 \\ . \\ . \\ . \\ \chi_\varrho \end{bmatrix} , \quad \dots (2.3)$$

Eq. (2.3) can be rewritten as:

$$B_\varrho(\omega) = [1 \;\; \omega \;\; \omega^2 \dots \omega^\varrho] . \begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \dots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix} . \begin{bmatrix} \chi_0 \\ \chi_1 \\ . \\ . \\ . \\ \chi_\varrho \end{bmatrix} , \quad \dots (2.4)$$

where $\sigma_{\upsilon\upsilon}$ $(\upsilon = 0, 1, \dots, \varrho)$ denotes the known parameters used to compute Touchard parameters. In this case, the matrix is square and non- singular. Where $\sigma_{\upsilon\upsilon}$ (υ=0, 1, 2, … , ϱ) are the known parameters that are utilized to calculate Touchard parameters. The first derivative of Eq. (2.1) is:

$$\psi'_\varrho(\omega) = \frac{d}{d\omega} \sum_{\upsilon=0}^{\varrho} \phi(\varrho, \upsilon)\omega^\upsilon = \sum_{\upsilon=0}^{\varrho} \binom{\varrho}{\upsilon} \upsilon\, \omega^{\upsilon-1} , \binom{\varrho}{\upsilon} = \frac{\varrho!}{\upsilon!\,(\varrho-\upsilon)!} , \quad \dots (2.5)$$

Also, Eq. (2.4) has the following derivative

$$B'_\varrho = [0 \;\; 1 \;\; 2\omega \dots \varrho\omega^{\varrho-1}] . \begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \dots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix} . \begin{bmatrix} \chi_0 \\ \chi_1 \\ . \\ . \\ . \\ \chi_\varrho \end{bmatrix} , \quad \dots (2.6)$$

3. Solution the (NLFID) equation

Since Eq. (1.1) has the form

$$\begin{cases} B'(\omega) = C(\omega) + \int_0^1 I(\omega,\varphi)K(B(\varphi))d\varphi, & \omega \in [d,e] \\ B(0) = b_0, \end{cases} \quad \ldots (3.1)$$

assume the following using Eqs. (2.4) and (2.6):

$$B(\omega) \cong B_\varrho(\omega) = [1 \;\; \omega \;\; \omega^2 \ldots \omega^\varrho].\begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \ldots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix}.\begin{bmatrix} \chi_0 \\ \chi_1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_\varrho \end{bmatrix}, \quad \ldots (3.2)$$

and

$$B'(\omega) \cong B'_\varrho = [0 \;\; 1 \;\; 2\omega \ldots \varrho\omega^{\varrho-1}].\begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \ldots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix}.\begin{bmatrix} \chi_0 \\ \chi_1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_\varrho \end{bmatrix}, \quad \ldots (3.3)$$

Eqs. (3.2) and (3.3) are now substituted into Eq. (3.1), yielding

$$[0 \;\; 1 \;\; 2\omega \ldots \varrho\omega^{\varrho-1}].\begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \ldots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix}.\begin{bmatrix} \chi_0 \\ \chi_1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_\varrho \end{bmatrix} = C(\omega) +$$

$$= \int_0^1 I(\omega,\varphi)\,[1 \;\; \varphi \;\; \varphi^2 \ldots \varphi^\varrho].\begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \ldots & \sigma_{0\varrho} \\ 0 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\varrho} \\ 0 & 0 & \sigma_{22} & \cdots & \sigma_{2\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varrho\varrho} \end{bmatrix}.\begin{bmatrix} \chi_0 \\ \chi_1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_\varrho \end{bmatrix} d\varphi \quad \ldots (3.4)$$

Now, the integral [13 and 14] can be calculated by putting $\omega = \omega_\upsilon, (\upsilon = 0,1,\ldots,\varrho)$ in Eq. (3.4), where $\omega_\upsilon = d + \upsilon h$, and the spacing $h = \frac{e-d}{\varrho}$, a non- linear system of algebraic equations is produced as a result. The Newton repeating approach can be used to solve this system. The parameters $(\chi_0, \chi_1, \ldots, \chi_\varrho)$ are found, and then the approximate answer of Eq. (1.1) was determined by substituting into Eq. (2.2).

4. Method Convergence

The solution accuracy [15 and 16] of the (NLFID) equation of the 2nd type can be obtained it's as follows: The truncated (TPs) series must be basically satisfied Eq. (1.1), then, for every $\omega = \omega_\upsilon \in [d, e],\ \upsilon = 0, 1, 2, \ldots, \varrho$, the error functions:

$$\text{Err}\ (\omega_\upsilon) = \left| \left(\sum_{\upsilon=0}^{\varrho} \chi_\upsilon \psi_\upsilon(\omega_\upsilon) \right)' - C(\omega_\upsilon) - \int_0^1 I(\omega_\upsilon, \varphi) \sum_{t=0}^{\varrho} \omega_t \varphi_\upsilon(\varphi)\, d\varphi \right| \cong 0, \text{then},$$

the Err $(\omega_\upsilon) \leq \in$, for every ω_υ in [d, e] and $\in > 0$. The degree ϱ is increased until the Err (ω_υ) becomes small enough. The following equation can be used to determine the error function

$$\text{Err}\ (\omega) = \left(\sum_{\upsilon=0}^{\varrho} \chi_\upsilon \psi_\upsilon(\omega) \right)' - C(\omega) - \int_0^1 I(\omega, \varphi) \sum_{\upsilon=0}^{\varrho} \chi_\upsilon \psi_\upsilon(\varphi)\, d\varphi\ ,$$

$\text{Err}_\varrho\ (\omega) \to 0$ whenever ϱ is right. The error diminishes and the Err $(\omega) \leq \in$.

5. Numerical Examples

Three examples were discussed in this part to demonstrate the efficacy of the proposed strategy in this study.

Example1. Solve the (NLFID) equation, given in [4 and 17]

$$B'(\omega) = 1 - \frac{1}{4}\omega + \int_0^1 \omega\varphi\, (B(\varphi))^2\, d\varphi\,, \qquad 0 \leq \varphi \leq 1$$

where $B(0) = 0$, and $B(\omega) = \omega$ is an exact solution

By applying the suggested method for the degree ϱ= 1, and then solving the non-linear systems via Newton repeating approach and MATLAB R2018b, The result was to get parameters of this degree, which lead to an exact solution as follows:

$$B_1(\omega) = (-1)\,\psi_0(\omega) + (1)\,\psi_1(\omega) = \omega$$

The comparison reveals that the offered method produced the same analytic solution of this example. Using the approach of the integral mean value, [4], also obtained the same exact solution. But, [17] didn't get the exact solution for n=2, 3, 4 and 5, via Bernstein polynomial method. As a result, our technique was more exact than [17] and similar to [4] with similar precision. Figure1 demonstrates the comparison to the exact solution for ϱ= 1.

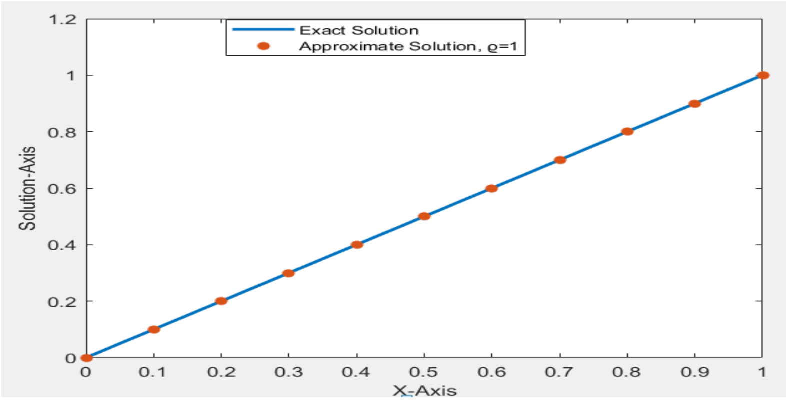

Figure1. Comparison for ϱ=1 of example 1

Example2. Solve the (NLFID) equation, given in [10, 17]

$$B'(\omega) = \frac{5}{4} - \frac{1}{3}\omega^2 + \int_0^1 (\omega^2 - \varphi)\,(B(\varphi))^2\, d\varphi\,, \qquad 0 \le \varphi \le 1$$

where $B(0) = 0$, and $B(\omega) = \omega$ is an exact solution

By applying the proposed method for the degree ϱ= 1, and then solving the non- linear systems via Newton repeating approach and MATLAB R2018b, The result is to get parameters of this degree, which lead to an exact solution as follows:

$$B_1(\omega) = (-1)\,\psi_0(\omega) + (1)\,\psi_1(\omega) = \omega$$

The comparison clears that the proposed method produced the same exact solution. The reference [17], used Bernstein approach, also obtained the same exact solution for n=2. But, [10] didn't get the exact solution for k=16, and 32, via the rationalized Haar function method. As a result, our technique was more exact than [10, 17]. Figure2 demonstrates the comparison to the exact solution for ϱ=1.

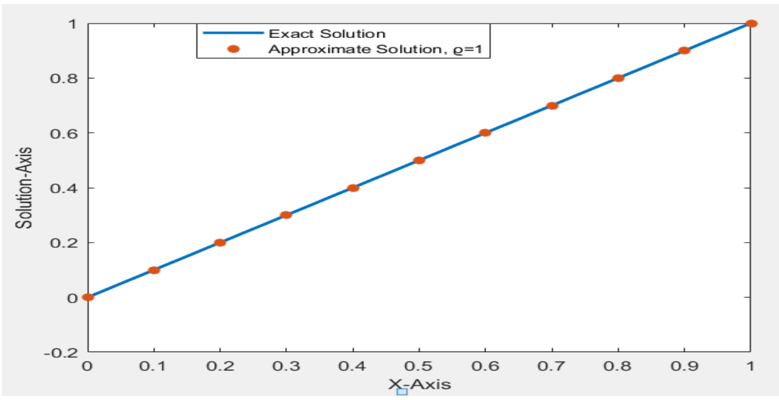

Figure2. Comparison for ϱ=1 of example 2

Example3. Solve the (NLFID) equation, given in [18, 19]

$$B'(\omega) = 1 - \frac{1}{3}\omega^3 + \int_0^1 \omega^3 \, (B(\varphi))^2 \, d\varphi \,, \qquad 0 \le \varphi \le 1$$

where $B(0) = 0$, and $B(\omega) = \omega$ is an exact solution.

By applying the suggested method for the degree ϱ= 1, 2, and 3, and then solving the non- linear systems via Newton repeating approach and MATLAB R2018b, The result is to get parameters of each degree, which lead to an exact solution as follows:

$$B_1(\omega) = (-1)\,\psi_0(\omega) + (1)\,\psi_1(\omega) = \omega$$
$$B_2(\omega) = (-1)\,\psi_0(\omega) + (1)\,\psi_1(\omega) + (0)\psi_2(\omega) = \omega$$
$$B_3(\omega) = (-1)\,\psi_0(\omega) + (1)\,\psi_1(\omega) + (0)\psi_2(\omega) + (0)\psi_3(\omega) = \omega$$

The reference [18] found the maximum errors for n=5, 9, 33 and 129 were $2.02 * 10^{-4}$, $5.02 * 10^{-5}$, $2.97 * 10^{-6}$ and $1.51 * 10^{-7}$ respectively for linear case, and $1.34 * 10^{-4}$, $1.71 * 10^{-5}$, $2.64 * 10^{-7}$ and $3.87 * 10^{-9}$ respectively for quadratic case. Also [19] get exact solution. Our approach was more effective than the previous one. For ϱ=1, 2 and 3, Figure3 shows the comparison with the exact solution for ϱ= 1, 2 and 3.

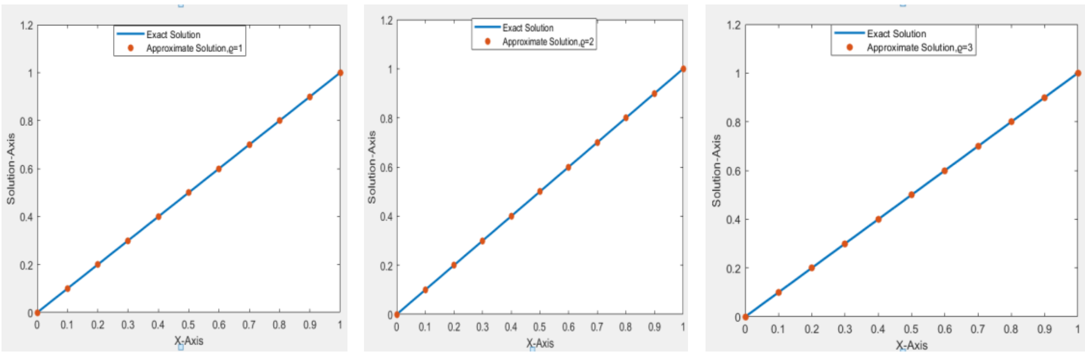

Figure3.a. Comparison for ϱ=1 **Figure3.b. Comparison for ϱ=2** **Figure3.c. Comparison for ϱ=3**

6. Conclusions and Recommendations

Analytically solving for the (NLFID) equations was frequently difficult. In many applications, the exact solutions are necessary, so, the proposed approach achieved what was in the ambitious in getting the exact solutions of the three examples to be solved. This approach may be extended to high-order non- linear integro- differential equations, as well as systems of the (NLFID) equations, with some adjustments.

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