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TOUCHARD METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATION OF SECOND TYPE

Jalil Talab ABDULLAH [1]

Wasit University, Iraq

Ali Hussein SHUAA [2]

Wasit University, Iraq

Hayder M AL-SAEDI [3]

University of Baghdad, Iraq

Abstract:

In this paper, a novel numerical method has been outlined for finding a novel solution to non-linear Volterra integral (NLVI) equation of second type, besides, two kernel type of this equation. Touchard polynomials (TPs) and to different degrees were used for this purpose. The function of approximation was obtained to derive the technique for solving this type of integral equations. Three numerical examples were provided to demonstrate the importance of the method used and the accuracy of the extracted results. In some examples, the results were compared with those of another method. The MATLAB R2018b program was used to carry out all calculations and generate all graphics.

Keywords: Nonlinear Volterra Integral Equation, Numerical Method, Touchard Polynomials, Exact Function, Numerical Solution.

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1 jabd@uowasit.edu.iq

2 alishuaa@uowasit.edu.iq

3 hayderma_math@uobaghdad.edu.iq

1. Introduction

Numerous scientific disciplines, including population dynamics, the spread of diseases, and semiconductor devices, are affected by nonlinear Volterra integral equations. Volterra began experimenting with integral equations in 1884, but his earnest research didn't start until 1896. Du Bois-Reymond gave the integral equation its name in 1888. Lalesco, however, was the one who gave the Volterra integral equation its name in 1908. (Abdul-Majid, W. A. Z., 2011). The (NLVI) equation enters many fields of theoretical and applied scientific knowledge such as: electrical applications in elasticity, plasticity, heat transfer, oscillation theory, fluid dynamics, control issue, and electrostatics (Polyanin, A. D.; Manzhirov, A. V., 2008). A great deal of researchers and scientists are interested in solving the (NLVI) equation to obtain numerical solutions for it via numerous efficient techniques has been proposed, these include: fourth order block by block method with Simpson's rule (Kasumo, C., ; Moyo, E., 2020), Bernstein polynomial method (Rani, D.; Mishra, V. 2019), the bound on a solution (Ramm, A. G., (2019), M-iteration method (Udo, M. O. et al.,2022), Taylor series expansion (Nadir, N. N., 2020), a successive approximation method (Maleknejad, K. et al., 2016), Variational iteration method (Al-Saar, F. M. and Ghadle, K. P., 2019), Newton-Raphson formula (Rani, D., and Mishra, V., 2018).

The standard form of the (NLVI) equation of 2nd type is (Abdul-Majid, W. A. Z., 2011)

$$\gamma(\varepsilon) = \beta(\varepsilon) + \int_0^{\varepsilon} \delta(\varepsilon, \rho)\, K(\gamma(\rho))\, d\rho \ , \qquad \varepsilon \in [x, c] \qquad \cdots (1.1)$$

for this type of equations, the kernel $\delta(\varepsilon, \rho)$ and the function $\gamma(\varepsilon)$ are given real valued functions and $K(\gamma(\varepsilon))$ is a nonlinear function for $\gamma(\varepsilon)$.

2. Method Description

Let's start by defining Touchard polynomials, which were the subject of study for French mathematician Jacques Touchard. A binomial-type polynomial sequence makes up Touchard polynomials, which are defined on [0, 1] (Nazir, A. et al., 2014; Abdullah, J. T., 2021) as follows:

$$S_\omega(\varepsilon) = \sum_{h=0}^{\omega} I(\varepsilon, h)\varepsilon^h = \sum_{h=0}^{\omega} \binom{\omega}{h} \varepsilon^h, \quad \binom{\omega}{h} = \frac{\omega!}{h!\,(\omega - h)!} \qquad \dots (2.1)$$

where the degree and index of the (TPs), respectively, are ω and h.

Below are the first five of these polynomials:

$S_0(\varepsilon)=1$

$S_1(\varepsilon)=1+\varepsilon$

$S_2(\varepsilon)=1+2\epsilon+\varepsilon^2$

$S_3(\varepsilon)=1+3\varepsilon+3\varepsilon^2+\varepsilon^3$

$S_4(\varepsilon)=1+4\varepsilon+6\varepsilon^2+4\varepsilon^3+\varepsilon^4$

2.1. Function of Approximation

The linear combination $\gamma_\omega(\varepsilon)$ of Touchard bases is defined as an approximate solution to Eq. (1.1) as follows:

$$\gamma_\omega(\varepsilon) = u_0 S_0(\varepsilon) + u_1 S_1(\varepsilon) + \cdots + u_\omega S_\omega(\varepsilon) = \sum_{h=0}^{\omega} u_h S_h\,(\varepsilon), \qquad 0 \leq \ \varepsilon \ \leq 1 \dots (2.2)$$

the function $\{S_h(\varepsilon)\}_{h=0}^{\omega}$ are Touchard basis of ω-th degree, as stated in Eq.(2.1), also u_h $(h = 0,1,\ldots,\omega)$ represents the unidentified Touchard parameters will be subsequently. Equation (2.2) written as a dot product:

$$\gamma_\omega(\varepsilon) = [S_0(\varepsilon) \;\; S_1(\varepsilon) \ldots S_\omega(\varepsilon)] \,.\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_\omega \end{bmatrix} \quad , \quad \ldots (2.3)$$

Eq.(2.3) is expressed as:

$$\gamma_\omega(\varepsilon) = [1 \;\; \varepsilon \;\; \varepsilon^2 \ldots \varepsilon^\omega].\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0\omega} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\omega} \\ 0 & 0 & \alpha_{22} & \cdots & \alpha_{2\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{\omega\omega} \end{bmatrix}.\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_\omega \end{bmatrix}, \ldots \quad (2.4)$$

where the known parameters α_{hh} $(h = 0,1,2,\ldots,\omega)$ are used to calculate Touchard coefficients. In this case, the matrix is square and non-singular.

3. Solution the (NLVI) equation of 2nd type

Given that Eq. (1.1) has the following form

$$\gamma(\varepsilon) = \beta(\varepsilon) + \int_0^\varepsilon \delta(\varepsilon,\rho)\, K(\gamma(\rho))\,d\rho \;, \qquad \cdots (3.1)$$

Using Eq. (2.4), suppose that:

$$\gamma(\varepsilon) = \gamma_\omega(\varepsilon) = [1 \;\; \varepsilon \;\; \varepsilon^2 \ldots \varepsilon^\omega].\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0\omega} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\omega} \\ 0 & 0 & \alpha_{22} & \cdots & \alpha_{2\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{\omega\omega} \end{bmatrix}.\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_\omega \end{bmatrix}, \ldots \quad (3.2)$$

Putting Eq. (3.2) into Eq. (3.1) produces the following

$$[1 \;\; \varepsilon \;\; \varepsilon^2 \ldots \varepsilon^\omega].\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0\omega} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\omega} \\ 0 & 0 & \alpha_{22} & \cdots & \alpha_{2\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{\omega\omega} \end{bmatrix}.\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_\omega \end{bmatrix}$$

$$= \beta(\varepsilon) + \int_0^\varepsilon \delta(\varepsilon,\rho)\,[1 \;\; \rho \;\; \rho^2 \ldots \rho^\omega].\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0\omega} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\omega} \\ 0 & 0 & \alpha_{22} & \cdots & \alpha_{2\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{\omega\omega} \end{bmatrix}.\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_\omega \end{bmatrix} d\rho \ldots (3.3)$$

then, the integral in Eq.(3.3) (Mustafa and AL Zubaidy, 2011; Zarnan, 2019) can be calculated by putting $\kappa = \kappa_\delta, (\,\delta = 0,1,\ldots,\omega)$ in Eq. (3.3), where $\kappa_\delta = r + \delta\lambda$, and spacing $\lambda = \frac{c-x}{\omega}$, Consequently, an algebraic equation system that is nonlinear is created. The solution to this system was using "Newton's iterative method". The coefficients $(u_0, u_1, \dots, u_\omega)$ are obtained, and then substituted into Eq. (2.2) to get the approximate numerical solution of Eq. (1.1)

4. Convergence Analysis:

The solution accuracy (Kurt et al., 2013; Biçer et al., 2018) of the (NLVD) equation of second type second type can be obtained as below: The truncated Touchard series in Eq. (2.2) must be basically satisfied Eq. (1.1). Consequently, for each $\varepsilon = \varepsilon_h \in [x, c], h = 0, 1, 2, \dots, \omega$, the error functions

$$\mathrm{Er}\,(\varepsilon_h) = \left| \sum_{h=0}^{\omega} u_h S_h\,(\varepsilon) - \beta(\varepsilon_h) - \int_0^{\varepsilon_h} \delta(\varepsilon_h, \rho) \sum_{h=0}^{\omega} u_h S_h(\rho)\, d\rho \right| \cong 0, \text{then},$$

the $\mathrm{Er}\,(\varepsilon_h) \leq \epsilon$, for each ε_h in the given interval [x, c] and $\epsilon > 0$, the truncation limit ω is then increased until the $\mathrm{Er}\,(\varepsilon_h)$ decrease in size enough. The error function can be calculated using the equation shown below:

$$\mathrm{Er}\,(\varepsilon) = \sum_{h=0}^{\omega} u_h S_h\,(\varepsilon) - \beta(\varepsilon) - \int_0^{\varepsilon} \delta(\varepsilon, \rho) \sum_{h=0}^{\omega} u_h S_h(\rho)\, d\rho\ ,$$

When $\mathrm{Er}_\omega\,(\varepsilon) \to 0$, the value of ω is sufficiently high, the error gradually decreases and then $\mathrm{Er}\,(\varepsilon) \leq \epsilon$.

5. Numerical Illustrations

The proposed approach is tested in this section by examining three examples of the (NLVI) equation. The accuracy of the solution approach was evaluated using the absolute error. The following was stated as the general formula:

Absolute error= $|\gamma(\varepsilon_h) - \gamma_\omega(\varepsilon_h)|$, $\varepsilon_{h\in[0,1]}$ and $h = 0, 1, \dots, \omega$ where $\gamma(\varepsilon_h)$ and $\gamma_\omega(\varepsilon_h)$ are the exact and approximate solutions of the (NLVI) equations, respectively.

Example1: Consider the (NLVI) equation given in (D. Rani, and V. Mishra, 2018).

$$\gamma(\varepsilon) = 2\varepsilon - (\frac{1}{12})\varepsilon^4 + 0.25 \int_0^{\varepsilon} (\varepsilon - \rho)\gamma^2(\rho)\, d\rho\,, \quad 0 \leq \varepsilon \leq 1$$

with the exact solution function is $\gamma(\varepsilon) = 2\,\varepsilon$. Applying the presented method and selecting the points $\varepsilon_0 = 0.1$ and $\varepsilon_1 = 0.2$ in the [0, 1]. The nonlinear system in Eq. (3.3) can be solving for ω= 2 by "Newton's iterative method" and "MATLAB R2018b" to get Touchard parameters, after substituting these parameters into Eq. (2.2). The approximate solution obtained is the same as the exact solution function given in the example which is as follows:

$$\gamma_2(\varepsilon) = -2S_0(\varepsilon) + 2S_1(\varepsilon) + (0)S_2(\varepsilon) = 2\,\varepsilon.$$

The reference (D. Rani, and V. Mishra, 2018) is obtained 0.003 as the maximum absolute error by using Newton-Raphson formula. As a result, our suggested method performs better. The comparison with the exact solution for ω=2 is shown in Figure1.

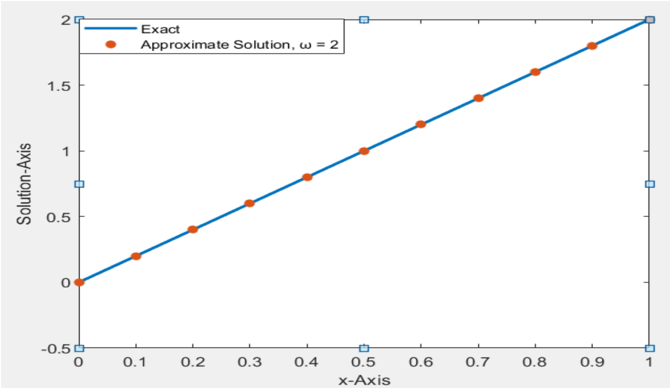

Figure1: Comparison of Solutions, ω=2 for Example1.

Example2: Consider the (NLVI) equation given in (H. Erfanian and T. Mostahsan, 2018)

$$\gamma(\varepsilon) = \exp(\varepsilon) - \frac{1}{2}(\exp(2\varepsilon) - 1) + \int_{0}^{\varepsilon} \gamma^2(\rho)\, d\rho \qquad 0 \le \varepsilon \le 1 \quad ,$$

with the exact solution function is $\gamma(\varepsilon) = \exp(\varepsilon)$.

Now, after applying the presented method for ω=2, 3 and 5, approximate numerical solutions are obtained respectively as follows:

$$\gamma_2(\varepsilon) = (6.4672e - 01)S_0(\varepsilon) + (-2.6738e - 01)S_1(\varepsilon) + (6.2207e - 01)\, S_2(\varepsilon)$$

$$\gamma_3(\varepsilon) = (2.8104e - 01)S_0(\varepsilon) + (6.4419e - 01)S_1(\varepsilon) + (-1.3290e - 01)S_2(\varepsilon) + (2.0762e - 01)S_3(\varepsilon).$$

$$\gamma_5(\varepsilon) = (2.5879e - 01)S_0(\varepsilon) + (9.5934e - 01)S_1(\varepsilon) + (-1.0496))S_2(\varepsilon) + (1.3031))S_3(\varepsilon) +$$
$$(-5.8881e - 01))S_4(\varepsilon) + (1.1720e - 01))S_5(\varepsilon)\ .$$

Tables 1 and 2 compare the approximate solutions and absolute error, respectively, demonstrating that the results' accuracy increases as ω increases. Also the comparison with the exact solution function for ω=2, 3, and 5 are shown in Figure2.

Table1: Exact & Approximate Solutions of Example2.

	Exact solution	Approximate solution, ω=2	Approximate solution, ω=3	Approximate solution, ω=5
0.0	1.0000e+00	1.0014e+00	9.9995e-01	1.0000e+00
0.1	1.1052e+00	1.1053e+00	1.1052e+00	1.1051e+00
0.2	1.2214e+00	1.2216e+00	1.2215e+00	1.2210e+00
0.3	1.3499e+00	1.3504e+00	1.3500e+00	1.3485e+00
0.4	1.4918e+00	1.4916e+00	1.4921e+00	1.4887e+00
0.5	1.6487e+00	1.6453e+00	1.6490e+00	1.6433e+00
0.6	1.8221e+00	1.8114e+00	1.8219e+00	1.8144e+00
0.7	2.0138e+00	1.9900e+00	2.0121e+00	2.0047e+00
0.8	2.2255e+00	2.1809e+00	2.2208e+00	2.2181e+00
0.9	2.4596e+00	2.3844e+00	2.4493e+00	2.4590e+00
1.0	2.7183e+00	2.6002e+00	2.6988e+00	2.7333e+00

Table2: Approximate Numerical Results of Example2.

ε	A B S O L U T E E R R O R S		
	ω=2	ω=3	ω=5
0.0	1.4100e-03	5.0000e-05	2.0000e-05
0.1	1.3578e-04	1.1302e-05	2.1767e-05
0.2	2.4204e-04	5.6602e-05	3.9727e-04
0.3	5.6549e-04	1.6833e-04	1.3850e-03
0.4	1.7950e-04	3.0658e-04	3.1111e-03
0.5	3.4138e-03	2.9623e-04	5.4219e-03
0.6	1.0708e-02	1.8728e-04	7.7573e-03
0.7	2.3796e-02	1.6336e-03	9.0260e-03
0.8	4.4598e-02	4.7151e-03	7.4819e-03
0.9	7.5232e-02	1.0306e-02	6.0298e-04
1.0	1.1804e-01	1.9502e-02	1.5028e-02

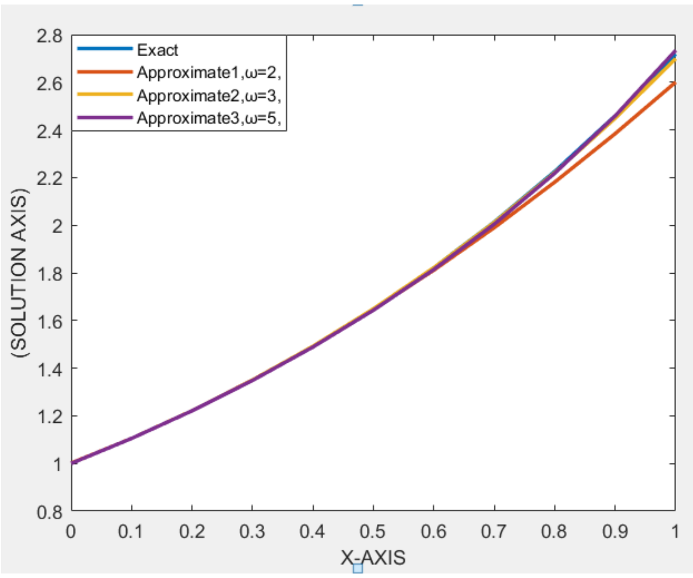

Figure2: Comparison of Solutions, ω=2, 3, 5 for Example2.

Example3: Consider the (NLVI) equation given in (D. Rani and V. Mishra, 2019):

$$\gamma(\varepsilon) = \frac{1}{4} + \frac{\varepsilon}{2} + \exp(\varepsilon) - \frac{\exp(2\varepsilon)}{4} + \int_0^{\varepsilon} (\varepsilon - \rho)\gamma^2(\rho)\, d\rho\,, \quad 0 \le \varepsilon \le 1$$

with the exact solution function is $\gamma(\varepsilon) = \exp(\varepsilon)$.

by applying the presented method for ω=2, and 3, approximate numerical solutions are obtained respectively as follows:

$$\gamma_2(\varepsilon) = (6.5283e-01)S_0(\varepsilon) + (-2.7915e-01)S_1(\varepsilon) + (6.2764e-01)\,S_2(\varepsilon)$$

$$\gamma_3(\varepsilon) = (2.1469e-01)S_0(\varepsilon) + (8.5120e-01)S_1(\varepsilon) + (\,-3.3836e-01)S_2(\varepsilon) + (2.7129e-01)S_3(\varepsilon)$$

Tables 3 and 4 compare the approximate solutions and absolute error, respectively, demonstrating that the results' accuracy increases as ω increases. Also the comparison with the exact solution function for ω=2, and 3 are shown in Figure3.

Table3: Exact & Approximate Solutions of Example3.

ε	Exact ion	Approximate solution, ω=2	Approximate solution, ω=3
0.0	1.0000e+00	1.0013e+00	9.9882e-01
0.1	1.1052e+00	1.1052e+00	1.1027e+00
0.2	1.2214e+00	1.2217e+00	1.2177e+00
0.3	1.3499e+00	1.3506e+00	1.3454e+00
0.4	1.4918e+00	1.4922e+00	1.4876e+00
0.5	1.6487e+00	1.6463e+00	1.6458e+00
0.6	1.8221e+00	1.8129e+00	1.8216e+00
0.7	2.0138e+00	1.9922e+00	2.0167e+00
0.8	2.2255e+00	2.1839e+00	2.2327e+00
0.9	2.4596e+00	2.3882e+00	2.4713e+00
1.0	2.7183e+00	2.6051e+00	2.7340e+00

Table4: Approximate Numerical Results of Example3.

ε	ABSOLUTE ERRORS	
	ω=2	ω=3
0.0	1.3200e-03	1.1800e-03
0.1	3.8482e-05	2.4895e-03
0.2	2.4884e-04	3.7220e-03
0.3	7.8779e-04	4.4131e-03
0.4	3.6970e-04	4.2205e-03
0.5	2.4263e-03	2.9375e-03
0.6	9.1704e-03	5.0656e-04
0.7	2.1598e-02	2.9647e-03
0.8	4.1627e-02	7.1860e-03
0.9	7.1378e-02	1.1665e-02
1.0	1.1319e-01	1.5688e-02

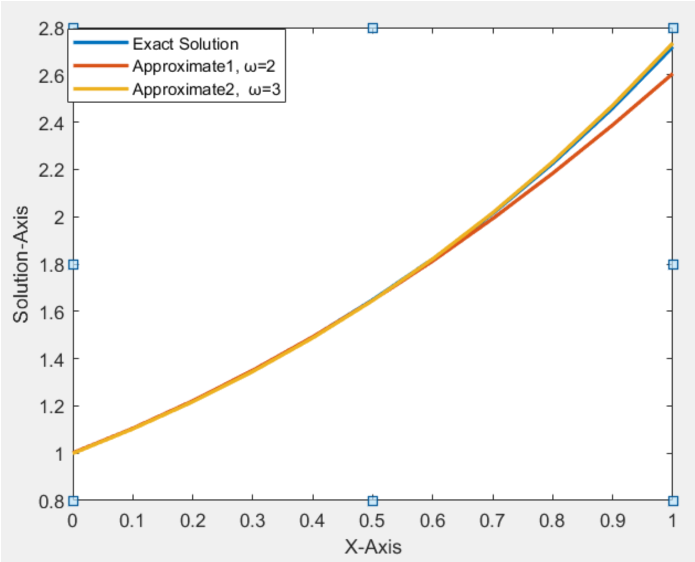

Figure3: Comparison of Solutions, ω=2, and 3 for Example3.

6. Conclusions

In this work, the approximate numerical solution of the (NLVI) equation was determined using the Toucard method. The results reached in this technique indicate that the presented method was effective. Comparing the approximate numerical results with the given exact solution function and another method shows that the results are very well corresponded to as shown in the tables and figures.

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