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T- HOLLOW-LIFTING MODULES

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Abstract:

Let there be a left-unitary Module over ring R which has identity called K. In this paper, we presented an introduction to the idea of T-hollow-lifting Module. An R -Module K is T-hollow lifting Module if, K/U T-hollow for each Submodule U of K , ∃ a Submodule V of K satisfies the following conditions:. K = V⊕V* and V⊆_Tce U in K In addition to this, we show it with various instances and describe some of its fundamental aspects.

Keywords: T-Small Submodule, T-Lifting Modules, T-Hollow Modules.

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Introduction:

Let K be $\mathbb{R}$-Module which has a Submodule T. A Submodule E of K is T-small Submodule of K ($E \ll_T K$), if T is subset of E+W for any Submodule W of K, then T is subset of W [1]. Let T be a Submodule of $\mathbb{R}$-Module K that is not zero. We say K is T-hollow Module if every Submodule Y of K such that $T \not\subseteq Y$ is T-small Submodule of K [2]. Let K be $\mathbb{R}$-Module and V , U be its Submodules such that $V \subset U \subset K$. If $\frac{U}{V} \ll_T \frac{K}{V}$, then V is called T-coessential Submodule of U in K ($V \subseteq_{Tce} U$ in K). T-hollow factor Module K is an R-Module if $\exists$ a Submodule G of K such that $\frac{K}{G}$ is T-hollow Module. An R-Module K is T-lifting Module if, for each Submodule E of K, $\exists$ a direct Summand F of K such that. $F \subseteq_{Tce} E$ in K [2] . We talked about T-hollow-lifting Module in this paper. We also list some simple properties.

An R-Module K is T-hollow-lifting if, for each Submodule W of K with $\frac{K}{W}$ T-hollow, $\exists$ a Submodule E of K such that. $K = E \oplus E^*$ and $E \subseteq_{Tce} W$ in K.

It is very evident that Z_4 as Z-Module is T-hollow-lifting.

Every Module that does not have T-hollow factor Module is T-hollow-lifting.

T-hollow-lifting does not apply to Z as Z-Module. To demonstrate this, suppose Z is T-hollow-lifting. Take the Submodule 4Z. Nevertheless, Z/4Z is T-hollow, and there is a direct summand U of Z such that $U \subseteq_{Tce} 4Z$ in Z. Given that Z is indecomposable, then $U = 0$ necessarily implies that $4Z \ll_T Z$ which is a contradiction.

Every T-lifting Module is T-hollow-lifting, in particular each semisimple or T-lifting Module is T-hollow-lifting. Take, for instance, Z_{p^∞} as Z-Module, where p is prime number. It cannot be said that the opposite is true. Take, for instance: Let's say that K is an indecomposable R-Module that doesn't have any T-hollow factor Modules. It is not difficult to demonstrate that K is T-hollow-lifting. Claim that K is not T-lifting. In order to demonstrate this, let's assume that K be T-lifting and E is proper Submodule of K. Nevertheless, as K be T-lifting, then $\exists$ a Submodule Y of K such that $Y \subseteq_{Tce} E$ in K, and $K = Y \oplus Y_1$ holds true for some $Y_1 \subseteq K$. Due to the fact that K is indecomposable Module, we have Y = 0 , thus $E \ll_T K$ and hence, K be T-hollow, which is a contradiction.

Proposition1 Let K be $\mathbb{R}$-Module. $K = K_1 \oplus K_2$, with K_1 and K_2 being T-hollow Modules. Then K is T-hollow lifting Module iff K is T-lifting Module.

Proof: Suppose K be T-hollow lifting Module and W be a Submodule of K, and let $\pi_1: K \to K_1$ and $\pi_2: K \to K_2$ be natural projections maps. If $\pi_1(W) \neq K_1$ and $\pi_2(W) \neq K_2$, then. After it, $\pi_1(W) \ll_T K_1$ and $\pi_2(W) \ll_T K_2$. Hence, we have, $\pi_1(W) \oplus \pi_2(W) \ll_T K_1 \oplus K_2$. Say that $W \subseteq \pi_1(W) \oplus \pi_2(W)$ and in order to demonstrate this, let $w \in W$. After this, $w \in K = K_1 \oplus K_2$ and as a result, $w = (k_1, k_2)$, where $k_1 \in K_1$ and $k_2 \in K_2$. Hence, $\pi_1(w) = \pi_1((k_1, k_2)) = k_1$ and $\pi_2(w) = \pi_2((k_1, k_2)) = k_2$. Hence, $w = (\pi_1(w), \pi_2(w))$, and since $W \subseteq \pi_1(W) \oplus \pi_2(W)$, this indicates that $W \ll_T K$. Hence, K is T-lifting Module. Since $\pi_1(W) = W_1$, we may deduce that $\pi_1(W) = \pi_1(K)$. It is evident that $K = W + K_2$. According to the second isomorphism theorem, $(W + K_2)/W$ is equivalent to $K_2/W \cap K_2$. But K_2 is T-hollow Module, then $K_2/W \cap K_2$ must be T-hollow implies K/W be T-hollow Module, since K is T-hollow-lifting, K must be T-lifting. For the converse is clear.

The previous proposition can be used to illustrate the following examples.

1. Consider the Module $Z_2 \oplus Z_4$, it is evident that both Z_2 and Z_4 as Z-Module are T-hollow Modules. Because $K = Z_2 \oplus Z_4$ is T-lifting, we may conclude that it is T-hollow-lifting.

2. Consider the Module $K = Z_2 \oplus Z_8$. It should come as no surprise that Z_2 and Z_8 as Z-Module are T-hollow Modules. It is not difficult to observe that the expression $K = Z_2 \oplus Z_8$ does not include T-lifting. According to proposition 1, K does not T-hollow lifting.

Proposition2 Let K be $\mathbb{R}$-Module, if K be T-hollow Module, then for each proper Submodule W of K, K/W must be T-hollow Module.

Proof: Clear.

Proposition3 Every T-hollow Module is indecomposable

Proof: Clear.

Proposition4 Let K be an R-Module. The following assertions are identical to one another:

1. K be T-hollow Module.
2. K/V be T-hollow Module, and $V \ll_T K$ for some Submodule V of K.
3. Each non-zero T-small factor Module of K is indecomposable.

Proof: 1⇒2 Let's say that K is T-hollow Module, and V be any proper Submodule of K; hence, $V \ll_T K$. As a result, according to proposition 2, K/V is T-hollow.

2⇒1 Let's say that $U \ll_T K$, and K/U be T-hollow. Let E be proper Submodule of K. Hence, E+U ≠ K, and consequently,(E+U)/U $\ll_T$ K/U . If we assume that T $\subseteq E + V$ and that V $\subseteq K$, then T/U ⊆ (E+V)/U = (E+U)/U + (V+U)/U. Since (E+U)/U $\ll_T$ K/U, we may deduce that T ⊆ V+U. Nevertheless, $U \ll_T K$ comes before T $\subseteq$ V. Therefore K be T-hollow Module.

1⇒3 Take K is T-hollow Module, and U is a non-zero T-small factor Module of K. Hence, if we apply prop.2 to this, we see that K/U is T-hollow. As a result, according to proposition 3, K/U is indecomposable.

3⇒1 Let V be a proper Submodule of K. Assume that $T \subseteq V + E$, where $E \subseteq K$. Thus, as shown by [3,] K/($V \cap E$) ≅ K/W⊕ K/E. As K/($V \cap E$) is indecomposable by [4], thus either K/V = 0 or K/E = 0. But V is a proper Submodule of K, thus $\frac{K}{V} \neq 0$ Hence, K/E = 0, and so $K \subseteq E$, then K is T-hollow Module.

Proposition5 Let K be an indecomposable Module. Then K is T-hollow-lifting Module iff K be T-hollow or K has no T-hollow factor Modules.

Proof: Assume K be T-hollow-lifting Module and has T-hollow factor Module Thus, ∃ a proper Submodule V of K such that K/V T-hollow. Nevertheless, K is T-hollow-lifting, there is a direct summand F of K such that $F \subseteq_{Tce} V$ in K. Since K be indecomposable Module, then F = 0; therefore, $V \ll_T K$. Then, according to proposition 4, K is T-hollow. The converse is clear.

Proposition6 Let K_1 , …. , K_n be Modules that don't have any T-hollow factors. Modules.Then K = $K_1 \oplus \cdots \oplus K_n$ is T-hollow-lifting.

Proof: Let W be a Submodule of K such that K/W is T-hollow. But K_1+W/W+⋯ +K_n+ W/W = K/W, ∃ I ∈{1,. . .,n} such that K_i+W/W be T-hollow. Thus, K_i has T-hollow factor Module, contradiction. Therefore K is T-hollow-lifting.

An example of T-hollow lifting Module which is not T-lifting Module may be found by looking at Proposition 5, which provides a starting point for this endeavor. In point of fact, it is obvious that every indecomposable Module K that does not have a T-hollow factor Module is a T-hollow-lifting Module, but it does not mean that it is a T-lifting Module. On the other hand, stipulate that U is any indecomposable Module that does not include a T-hollow factor Module, and stipulate that W is a semisimple Module. If D be a Submodule of K = U⊕W such that K/D has the property of being T-hollow, then we have either U+D = K or

W+D = K. Because U does not possess any T-hollow factor Modules and U+D/D≅U/U∩D, we may conclude that W+D=K. Yet W is semisimple. Hence, there is a Submodule Y of W such that W = Y⊕(W∩D). Hence, Y⊕D = K. As a result, D may be thought of as a direct summand of K. As a consequence of this, K is doing a T-hollow lifting. Evidently, K is not T-lifting (U is not T-hollow).

Proposition7 Let K be T-hollow-lifting Module, and let V , Q be Submodules of K such that K/Q T-hollow and $T \subseteq V + Q$, then there is a direct summand Y of K such that $T \subseteq \frac{V+Y}{Y}$ and $Y \subseteq_{Tce} Q$ in K.

Proof: Suppose that V and Q are Submodules of K such that. K/Q T-hollow. Nevertheless, K is a T-hollow-lifting Module, ∃ a direct summand Y of K such that $Y \subseteq_{Tce} Q$ in K. Yet, but $T \subseteq V + Q$, implies T ⊆(V+Q)/Y = (V+Y)/Y + Q/Y. Given that $Y \subseteq_{Tce} Q$ in K, we may deduce that T ⊆ (V+Y)/Y.

Assuming that H and D are both Submodules of the R-Module K, we may say that H is the T-supplement of D in K if K = H + D and $H \cap D \ll_T H$.

If every Submodule of an R-Module K has a T-supplement in K, then the R-Module is referred to as a T-supplemented Module.

An R-Module K is an amply T-supplemented if for any Submodules U and V of K with $U + V = K$, V includes a T-supplement of U in K.

A Submodule U of an $\mathbb{R}$-Module K is said to be T-coclosed of K ($U \subseteq_{Tcc} K$) if $U/V \ll_T K/V$ implies that U = V for every V ≤ K contained in U.

Let K be an $\mathbb{R}$-Module and a Submodule H of K. If a Submodule F of H is both T-coessential Submodule of H in K and T-coclosed Submodule of K, thus F is referred to as T-coclosure Submodule of H in K. That is, $H/F \ll_T K/F$ and whenever D ⊆ F with $F/D \ll_T K/D$ implies that D = F.

Proposition8 Let K be amply T-supplemented Module. Then each Submodule of K has T-coclosure Submodule.

Proof: Assuming that U is a Submodule of K. Yet, K be amply T-supplemented; hence, ∃ a Submodule W of K such that. W is T-minimal, with the property K = U+W. Nonetheless, K is amply T-supplemented, ∃ a Submodule Y of K such that $Y \subseteq U$, $K = Y + W$ and $Y \cap W \ll_T Y$. To show Y is a T-coclosure Submodule of U in K. We must show $Y \subseteq_{Tce} U$ in

K. Let $Y \subseteq P \subset K$ such that $T \subseteq U/Y + P/Y$. By modular law, $P = P\cap(Y + W) = Y + (P\cap W)$, therefore $T \subseteq U/Y + (Y + (P\cap W))/Y$. This implies that $T \subseteq U/Y + (P\cap W)/Y$, and hence By minimality of W, we get $W = P \cap W$. So $T \subseteq U/Y + ((P\cap W))/Y = U/Y + P/Y$, we may conclude that $T \subseteq P/Y$. Hence, $Y \subseteq_{Tce} U$ in K.

Proposition9 Let K be an R-Module, and let U , V be Submodules of K such that $U \subset V \subset K$, If $U \subseteq_{Tce} V$ in K and K/V is T-hollow Module, then K/U is T-hollow Module

Proof: Suppose that $U \subseteq_{Tce} V$ in K and that K/V is T-hollow Module. According to the third theorem of isomorphism, $K/V \cong (K/U)/(V/U)$. Yet since K/V is T-hollow and $U \subseteq_{Tce} V$ in K, and hence by proposition 4 , K/U is T-hollow.

Proposition10 Let K be T-hollow-lifting Module, then every T-coclosed Submodule F of K with $\frac{K}{F}$ T-hollow is a direct summand of K.

Proof: Assume K be T-hollow-lifting Module, and F be T-coclosed Submodule in K such that $\frac{K}{F}$ T-hollow. But K be T-hollow-lifting, thus there is a direct summand W of K such that $W \subseteq_{Tce} F$ in K. Since F be T-coclosed in K, then $F = W$and heence F is a direct summand of K.

Proposition11 If K is an amply T-supplemented Module and every T-coclosed Submodule F of K such that K/F T-hollow is a direct summand of K, then K is T-hollow-lifting Module.

Proof: Suppose that K be amply T-supplemented Module and any T-coclosed Submodule F of K with $\frac{K}{F}$ T-hollow which is a direct summand. To prove K is T-hollow-lifting, let H Is a Submodule of K with $\frac{K}{H}$ T-hollow. As a result, with prop.8, H has a T-coclosure Submodule F in K. Then, $F \subseteq_{Tce} H$ in K and $F \subseteq_{Tcc} K$. Yet, since K/H is a T-hollow, according to prop.9, K/F must also be a T-hollow. Hence, F is a direct summand. Therefore, K is T-hollow-lifting.

Proposition12 Let K be R-Module and $E \subseteq K$. If E be T-supplement Submodule of K then E is T-coclosed Submodule of K.

Proof: Suppose that E is T-supplement of U in K. Hence, K= E+U, and H is the T-minimal. Let $W \subseteq E \subseteq K$ such that $E/W \ll_T K/W$. So $K/W = (E+U)/W = E/W + (U+W)/W$. Hence, $(U+W)/W = K/W$, and consequently, $K = U + W$. We have $W = E$ from the minimality of E. Hence, E is a T-coclosed of K.

An R-Module K is said to be a weakly T-supplemented Module if, for every Submodule H of K, there exists a Submodule F of K such that. K = H+F and $F \cap H \ll_T K$.

Proposition13 Let K be weakly T-supplemented Module and let $U \subseteq K$. If for all $D \subseteq K$ with $D \subseteq U$, $D \ll_T K$ implies $D \ll_T U$. then U is T-supplement Submodule of K.

Proof: Assume that K is weakly T-supplemented Module. Hence, ∃ a Submodule V of K may be expressed as K = U + V and $U \cap V \ll_T K$. According to our assumption, $U \cap V \ll_T U$.. Therefore U is T-supplement of V in K.

An ℝ-Module K is said to have property (D3) if, for any direct summand V and Y of K, where K = V + Y, $V \cap Y$ is a direct summand of K [5].

If both Submodule V and V- are T-supplements of each other, then V and Y are mutual T-supplements in ℝ-Module K,

Proposition14 Let W and E are mutual T-supplements in K such that $K = W + E$ be T-hollow-lifting Module, with K/W and K/E are T-hollow Modules. If K has (D3), then $K = W \oplus E$.

Proof: Suppose that Submodules W and E are mutual T-supplements in K, and that K/W , K/E are T-hollow Modules , then according to proposition 12, W and E are T-coclosed Submodules of K. But K be T-hollow-lifting, it follows that W and E are direct summands of K according to prop.10. Therefore, since K = W + E and K has (D3), then $W \cap E$ be direct summand of W, and $W = (W \cap E) \oplus D$, for some $D \subseteq K$. Nevertheless, as E is a T-supplement of W, $W \cap E \ll_T E$ and thus $W \cap E \ll_T K$ Hence, K = D, and $W \cap E = 0$. Then we obtain K = W⊕E.

Proposition15 Let K be a T-hollow-lifting Module having (D3). Then every direct summand of K is T-hollow-lifting.

Proof: Assume that Y is a direct summand of K. So K = Y ⊕ Y^* for any Submodule Y^* of K. Let E ≤ W such that Y/E is T-hollow. Therefore, (Y ⊕ Y^*)/E = Y/E ⊕ (Y^*⊕E)/E., According to [6, corr.3,44], K/E/(Y*⊕E)/E ≅ Y/E; hence, according to the third isomorphism theorem, K/E/(Y*⊕E)/E ≅K/(Y*⊕E). Yet, Y/U is T-hollow, and because of this, K/(Y*⊕E) is T-hollow. Yet, K is T-hollow-lifting, ∃ a direct summand V of K such that $V \subseteq_{ce}$ (Y^*⊕E) in K. Therefore, K/V = (Y ⊕ Y*)/V = (Y+V)/V + (Y^*+V)/V. Make the assertion that K ≠ Y^*+ V (since if K = Y*+ V, then K = Y*+ E, which is a contradiction). But, according to prop.7, K/V is T-hollow, hence K/V = (Y+V)/V. Then, K = Y+V. Thus, according to proposition 2, we obtain Y∩(Y^*⊕E)/(V∩Y) ≪ K/(V∩Y). This suggests that $V \cap Y \subseteq_{ce} E$ in K. Yet, since K has (D3), then V∩Y is a direct summand of K and V∩Y is a direct summand of Y. Yet, E/(V∩Y) ≤ W/(V∩Y) and Y/(V∩Y) is a direct summand of K/(V∩Y), the implication of proposition 10 V∩Y$\subseteq_{ce}$ E in Y. Therefore Y is T-hollow-lifting.

Let K be an ℝ-Module. A Submodule Y of K is fully invariant Submodule if $h(Y) \subseteq Y$, for every $h \in Hom(K,K)$[7].

Lemma16 [8] Let K be an ℝ-Module . If $K = K_1 \oplus K_2$, then $\frac{K}{Y} = \frac{Y+K_1}{Y} \oplus \frac{Y+K_2}{Y}$, for every fully invariant Submodule Y of K.

Proposition17 Let K is an R-Module that is capable of T-hollow-lifting, then we can say that K/U is capable of T-hollow-lifting for any fully invariant Submodule U of K.

Proof: Assuming that V/U is a Submodule of K/U such that (K/U)/(V/U) is T-hollow. Hence, according to the third theorem of isomorphism, (K/U)/(V/U) ≅ K/V is T-hollow. Yet, K is T-hollow-lifting Module; hence, ∃ a Submodule E of K satisfies $E \subseteq_{Tce} V$ in K, and $= E \oplus E^*$ for some $E^* \subseteq K$. It is obvious that $E + U \subset V$, and as a consequence, (E+U)/U ⊂ V/U. Let f : K/E→ K/(E+U) defined as f(d+E) = d+(E+U), ∀ d ∈ K.. Clearly that f is an epimorphism. . Yet, But $E \subseteq_{Tce} V$ in K, hence, f(V/E) $\ll_T$K/(E+U); consequently, $E + U \subseteq_{Tce} V$ in K. Thus, according to the third theorem of isomorphism, (E+U)/U $\subseteq_{Tce}$ V/U in K/U. By Lemma16, K/U = (E⊕E*)/U = (E+U)/U ⊕ (E*+U)/U. As a result, (E+U)/U is a direct summand of K/U. Then K/U is T-hollow-lifting.

An ℝ-Module K is duo-Module if each Submodule of K is fully invariant [8].

Proposition18 Each direct summand of the duo T-hollow-lifting Module K is T-hollow-lifting.

Proof: Clear, according to proposition 17

Theorem19 Let R is a commutative ring and that K is a non-zero indecomposable module over R. Hence, the following are equivalent:

1. Kis T-hollow-lifting.

2. Kis T-lifting.

3. Kis T-hollow.

Proof: Clear.

Lemma20 [3] Let $f: K \to H$ be an epimorphism of $\mathbb{R}$-Modules and $K = E + D$, where E and D are Submodules of K then:

1. $H = f(E) + f(D)$.
2. If $\ker f = E \cap D$, then $H = f(E) \oplus f(D)$.

Proposition21 Epimorphic image of T-hollow Module is T-hollow

Proposition22 Let $f: K \to U$ be an epimorphism of $\mathbb{R}$-Modules. Let Y and V be Submodules of K such that $K = Y + V$ and $\ker f = Y \cap V$. If U is T-hollow-lifting Module and V is T-hollow, then $U = K_1 \oplus K_2$, where $K_1 \subseteq_{Tce} f(Y)$ in U and K_2 is T-hollow.

Proof: By lemma20, U = (Y) ⊕ f (V). Yet, since V is T-hollow, according to proposition 21, f(V) must also be T-hollow,. According to the second theorem of isomorphism, U/(f(Y)) ≅ f(V). Then, U/(f(Y)) is T-hollow. Nevertheless, U is T-hollow-lifting Module; hence, ∃ a direct summand K_1 of U such that $K_1 \subseteq_{Tce} f(Y)$ in U. Hence, U = $K_1 \oplus K_2$ where $K_2 \subseteq U$. Hence, U/ K_1 = (f(Y) ⊕f(V))/ K_1 = (f(Y))/ K_1 + (f(V)⊕ K_1)/ K_1. This suggests that U = f (V) ⊕ K_1. According to the second isomorphism theorem, U/ K_1 ≅ f(V), and U/ K_1 ≅ K_2, this means that K_2 ≅ f(V), and hence K_2 is a T-hollow.

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