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STUDY ON NEW SUBFAMILY OF CONVEX FUNCTIONS ATTACHED SQUEAKY COEFFICIENT BOUNDS WITH TANH FUNCTION

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Abstract

We are emplying tanh function of family D_ϱ^ε of convex functions in Ʊ be introduced and investigated. The major support of this paper deals the derivations of squeaky inequalities which involve Taylor- Maclaurin cofficients for functions that attributed to class 〖D〗_ϱ^ε of convex function in Ʊ. In private, the bounds of first three coefficients of Taylor- Maclaurin series and the estimates of second-and third-order Hankel determinant say, HD_2,2 (f) and HD_3,1 (f) respectively and the estimates of the Fekete-Szegö functionals are the major aim for our study in this paper**.**

Keywords: *Second-And Third-Order Hankel Determinant, Univalent Functions ,Subordination , Convex Function, The Quantum Or Basic(Or Q -) Calculus And Its Trivlal(P, Q)-Variation, Tanh Function, Analytic Functions.*

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Introduction

Suppose we have a functions f with class $\wp$,where f is analytic function and attributed to open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, which takes the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, z \in \mathrm{U}. \qquad (1.1)$$

Suppose we have a univalent function of $\wp$, assume ϱ is subclass of $\wp$,and assume $\mathcal{F}$ is analytic function (L) class where its normalized is given by:

$$L(z) = 1 + \mathbb{S}_1 z + \mathbb{S}_2 z^2 + \mathbb{S}_3 z^3 + \cdots , \qquad (1.2)$$

and holds the below inequality

$$Re\big(L(z)\big) > 0, z \in \mathrm{U}.$$

Suppose we have two functions f *and* g, where both are analytic function in U .

Therefore, g is subordinate of the function f, which is given by

$f(z) \prec g(z), z \in \mathrm{U}$, if the Schwarz function $w(z)$ exists, with $|w(z)| < 1$ and $w(0) = 0$ [see more [21]] , $g(z) = f\big(w(z)\big), z \in \mathrm{U}$.

We introduced the new class.

Definition (1.1): suppose that the function $f \in \wp$ that taken the formula (1.1) is called to be new subfamily say class $\mathfrak{D}_\varrho^\varepsilon$ involving convex function:

$$\mathfrak{D}_\varrho^\varepsilon = \left\{1 + \frac{zf''(z)}{f'(z)} \prec 1 + tanh\, z\ , f \in \wp, z \in \mathrm{U}\right\}. \qquad (1.3)$$

In the other words, we present the hyperbolic function:

$\psi(z) = 1 + tanhz\, , \psi(0) = 1$ and $Re(\psi(z)) > 0$. $f \in \mathfrak{D}_\varrho^\varepsilon$ iff there exists a holomorphic function $\mathfrak{q}$, $\mathfrak{q}(z) \prec \mathfrak{q}_0(z) = 1 + tanhz$, where

$$f = ze^{\left(\int_0^z \frac{\mathfrak{q}(t)-1}{t} dt\right)}, \qquad (1.4)$$

consider the $\mathfrak{q}(z) = \mathfrak{q}_0(z) = 1 + tanhz$ we get from (1.4) thective a function of the extremal function which is taken in many problems of the class $\mathfrak{D}_\varrho^\varepsilon$, is given by:

$$f_0 = ze^{\left(\int_0^z \frac{tanht}{t} dt\right)}$$

$$= z + z^2 + \frac{z^3}{2} + \frac{z^4}{18} + \cdots.$$

The $\mathfrak{q}^{th}$ Hankel determinant had been stated by Noonan and Thomas [23] in 1976 where $\mathfrak{q} \geq 1$ and $\mathfrak{n} \geq 1$ of functions f given below.

$$\mathbb{HD}_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \ a_1 = 1 \ .$$

In particular, we have

For $q = 2, n = 1$ and $a_1 = 1$, $\mathbb{HD}_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. $\mathbb{HD}_{2,2}(f) = a_2 a_4 - a_3^2$ for $q = 2$, $n = 2$ denoted by the second Hankel and study the two classes bi-(convex and starlike) functions (see[1,2,3,4,5,9,10,12,15,16,25,31]). The third Hankel determinant is given as:

$$\mathbb{HD}_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \ q_1 = 3, n = 1 \ ,$$

where elements of the previous determinant are different classes of analytic functions which equivalent [6,7,8,11,12,13,17,18,20,22,27,28,29,30,32,33,34].

2. Preliminaries:

Lemma(2.1)[19]: Suppose $\mathfrak{B}(z) \in \wp$ **,** thus there exists some $z, \mathcal{X}$ together with , $|\mathcal{X}| \leq 1$, $|z| \leq 1$,such that

$$2\mathbb{S}_2 = \mathbb{S}_1^2 + \mathcal{X}(4 - \mathbb{S}_1^2).$$

$$4\mathbb{S}_3 = \mathbb{S}_1^3 + 2\mathbb{S}_1\mathcal{X}(4 - \mathbb{S}_1^2) - (4 - \mathbb{S}_1^2)\mathbb{S}_1\mathcal{X}^2 + 2(4 - \mathbb{S}_1^2)(1 - |\mathcal{X}|)z \ .$$

and

$$8\mathbb{S}_4 = \mathbb{S}_1^4 + \mathcal{X}[\mathbb{S}_1^2(\mathcal{X}^2 - 3\mathcal{X} + 3) + 4\mathcal{X}](4 - \mathbb{S}_1^2) - 4(4 - \mathbb{S}_1^2)(1 - |\mathcal{X}|^2)\mathcal{X}\big[\mathbb{S}(\mathcal{X} - 1)\xi + \bar{\mathcal{X}}\bar{\xi} - (1 - |\xi|)\zeta\big](4 - \mathbb{S}_1^2)$$

Lemma(2.2) [26]: Suppose $\mathfrak{B}(z) \in \wp$. Thus

$$|\mathbb{S}_1^4 + \mathbb{S}_2^2 + 2\mathbb{S}_1 v_3 - 3\mathbb{S}_1^2\mathbb{S}_2 - \mathbb{S}_4| \leq 2,$$

$$|\mathbb{S}_1^5 + 3\mathbb{S}_1\mathbb{S}_2^2 + 3\mathbb{S}_1^2\mathbb{S}_3 - 4\mathbb{S}_1^3\mathbb{S}_2 - 2\mathbb{S}_1\mathbb{S}_4 - 2\mathbb{S}_2\mathbb{S}_3 + \mathbb{S}_5| \leq 2 \ ,$$

$$|\mathbb{S}_1^6 + 6\mathbb{S}_1^2\mathbb{S}_2^2 + 4\mathbb{S}_1^3\mathbb{S}_3 + 2\mathbb{S}_1\mathbb{S}_5 + 2\mathbb{S}_2\mathbb{S}_4 + \mathbb{S}_3^2 - \mathbb{S}_2^3 - 5\mathbb{S}_1^4\mathbb{S}_2 - 3\mathbb{S}_1^2\mathbb{S}_4 - 6\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3 - \mathbb{S}_6| \leq 2 \ ,$$

$|\mathbb{S}_n| \leq 2$ where $n = 1,2,3, \ldots$.

Lemma (2.3) [24]: Suppose $\mathfrak{B}(z) \in \wp$, thus

$$\left|\mathbb{S}_2 - \frac{\mathbb{S}_1^2}{2}\right| \leq 2 - \frac{|\mathbb{S}_1|^2}{2} \ ,$$

$$|\mathbb{S}_{n+k} - \zeta\mathbb{S}_n\mathbb{S}_k| \leq 2 \ , \ 0 \leq \zeta \leq 1 \ ,$$

$$|\mathbb{S}_{n+2k} - \zeta\mathbb{S}_n\mathbb{S}_k^2| \leq 2(1 + 2\zeta) \ .$$

3. Main Results :

Now we present the prove and statement to our theorems as a part of our work.

Theorem (3.1): Suppose that $f \in \mathfrak{D}_{\varrho}^{\varepsilon}$ and be taken formula (1.1), thus

$$|\mathfrak{a}_2| \leq \frac{1}{2} \ , |\mathfrak{a}_3| \leq \frac{1}{6} \ , |\mathfrak{a}_4| \leq \frac{1}{12} \ . \qquad (3.1)$$

Proof : Suppose $f \in \mathfrak{D}_{\varrho}^{\varepsilon}$ and $\frac{zf''(z)}{f'(z)} = 1 + tanh(w(z))$, such that,

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2\mathfrak{a}_2 z + (6\mathfrak{a}_3 - 4\mathfrak{a}_2^2)z^2 + (12\mathfrak{a}_4 - 18\mathfrak{a}_2\mathfrak{a}_3 + 8\mathfrak{a}_2^3)z^3 +$$

$$(20\mathfrak{a}_5 - 18\mathfrak{a}_3^2 - 32\mathfrak{a}_2\mathfrak{a}_4 + 48\mathfrak{a}_2^2\mathfrak{a}_3 - 16\mathfrak{a}_2^4)\mathfrak{a}^4 \ . \qquad (3.2)$$

Suppose $\mathfrak{p}(z) \in \mathbb{p}(z)$,in the some conditions for Schwarz function $w(z)$,

$$\mathfrak{p}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \mathbb{S}_1 z + \mathbb{S}_2 z^2 + \mathbb{S}_3 z^3 + \cdots,$$

consider that $\mathbb{p}(z) \in \mathcal{F}$

and

$$w(z) = \frac{\mathfrak{p}(z)-1}{1+\mathfrak{p}(z)} = \frac{\mathbb{S}_1 z+\mathbb{S}_2 z^2+\mathbb{S}_3 z^3+\cdots}{2+1+\mathbb{S}_1 z+\mathbb{S}_2 z^2+\mathbb{S}_3 z^3+\cdots} \ .$$

On the other side,

$$1 + tanh(w(z)) = 1 + \frac{1}{2}\mathbb{S}_1 z + \left(\frac{\mathbb{S}_2}{2} - \frac{\mathbb{S}_1^2}{4}\right)z^2 + \left(\frac{\mathbb{S}_1^3}{12} - \frac{\mathbb{S}_1\mathbb{S}_2}{2} + \frac{\mathbb{S}_3}{2}\right)z^3 + \left(\frac{\mathbb{S}_4}{2} + \frac{5\mathbb{S}_1^2\mathbb{S}_2}{16}\right.$$

$$\left. + \frac{\mathbb{S}_1^2\mathbb{S}_2}{4} - \frac{\mathbb{S}_1\mathbb{S}_3}{2} - \frac{\mathbb{S}_2^2}{4}\right)z^4 \ . \qquad (3.3)$$

Comparing the coefficients of $z, z^2, \ldots, z^6$ between equations (3.2) and (3.3) and by using lemma (2.2) we have :

$$|\mathfrak{a}_2| \leq \frac{1}{2}$$

$$|\mathfrak{a}_3| \leq \frac{1}{6},$$

$$|\mathfrak{a}_4| = \left|\frac{\mathbb{S}_3}{24} - \frac{\mathbb{S}_1^3}{288} - \frac{\mathbb{S}_1\mathbb{S}_2}{96}\right| = \frac{\mathbf{1}}{\mathbf{24}}\left[\mathbb{S}_3 - \frac{\mathbb{S}_1\mathbb{S}_2}{4}\right] + \frac{\mathbb{S}_1}{\mathbf{144}}\left[\mathbb{S}_2 - \frac{\mathbb{S}_1^2}{4}\right].$$

$$|\mathfrak{a}_4| \leq \frac{1}{12} + \frac{\mathbb{S}\left(2 - \frac{\mathbb{S}_1^2}{2}\right)}{144}.$$

Also ,let

$\pi(\mathbb{S}) = \frac{1}{12} + \frac{\left(2\mathbb{S} - \frac{\mathbb{S}_1^3}{2}\right)}{144}$.

$\pi'(\mathbb{S}) = \frac{1}{72} + \frac{3\mathbb{S}^2}{288}$.

Put, $\pi'(\mathbb{S}) = 0$,we have $\mathbb{S} = \frac{2}{\sqrt{3}}$ and so $\pi(\mathbb{S})$ has a maximum value attained at $\mathbb{S} = \frac{2}{\sqrt{3}}$,also which is

$|\mathfrak{a}_4| \leq \pi\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{12}$.

Theorem (3.2): Suppose that the function $f \in \mathfrak{D}_\varrho^\varepsilon$ and be taking by (1.1), then :

$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{12}$.

Proof : suppose $f \in \mathfrak{D}_\varrho^\varepsilon$ and $\frac{zf''(z)}{f'(z)} = 1 + tanh(w(z))$, such that,

$1 + \frac{zf''(z)}{f'(z)} = 1 + 2\mathfrak{a}_2 z + (6\mathfrak{a}_3 - 4\mathfrak{a}_2^2)z^2 + (12\mathfrak{a}_4 - 18\mathfrak{a}_2\mathfrak{a}_3 + 8\mathfrak{a}_2^3)z^3 +$

$(20\mathfrak{a}_5 - 18\mathfrak{a}_3^2 - 32\mathfrak{a}_2\mathfrak{a}_4 + 48\mathfrak{a}_2^2\mathfrak{a}_3 - 16\mathfrak{a}_2^4)\mathfrak{a}^4$. (3.4)

Suppose $\mathfrak{p}(z) \in \mathbb{p}(z)$,in the some conditions for Schwarz function $w(z)$,

$$\mathfrak{p}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \mathbb{S}_1 z + \mathbb{S}_2 z^2 + \mathbb{S}_3 z^3 + \cdots,$$

consider $\mathbb{p}(z) \in \mathcal{F}$

and

$w(z) = \frac{\mathbb{p}(z)-1}{1+\mathbb{p}(z)} = \frac{\mathbb{S}_1 z + \mathbb{S}_2 z^2 + \mathbb{S}_3 z^3 + \cdots}{2+1+\mathbb{S}_1 z + \mathbb{S}_2 z^2 + \mathbb{S}_3 z^3 + \cdots}$.

On the other side,

$1 + tanh(w(z)) = 1 + \frac{1}{2}\mathbb{S}_1 z + \left(\frac{\mathbb{S}_2}{2} - \frac{\mathbb{S}_1^2}{4}\right)z^2 + \left(\frac{\mathbb{S}_1^3}{12} - \frac{\mathbb{S}_1\mathbb{S}_2}{2} + \frac{\mathbb{S}_3}{2}\right)z^3 + \left(\frac{\mathbb{S}_4}{2} + \frac{5\mathbb{S}_1^2\mathbb{S}_2}{16}\right.$

$\left. + \frac{\mathbb{S}_1^2\mathbb{S}_2}{4} - \frac{\mathbb{S}_1\mathbb{S}_3}{2} - \frac{\mathbb{S}_2^2}{4}\right)z^4$. (3.5)

Comparing the coefficients of $z, z^2, \ldots, z^6$ between equations (3.4) and (3.5), we have

$2\mathfrak{a}_2 = \frac{1}{2}\mathbb{S}_1$, (3.6)

$\mathfrak{a}_3 = \frac{1}{12}\mathbb{S}_2$, (3.7)

$\mathfrak{a}_4 = \frac{\mathbb{S}_3}{24} - \frac{\mathbb{S}_1^3}{288} - \frac{\mathbb{S}_1\mathbb{S}_2}{96}$, (3.8)

$$\mathfrak{a}_5 = \frac{\mathbb{S}_4}{40} - \frac{\mathbb{S}_1^2\mathbb{S}_2}{240} - \frac{\mathbb{S}_1\mathbb{S}_2}{120} - \frac{\mathbb{S}_2^2}{160} + \frac{\mathbb{S}_1^4}{576}. \quad (3.9)$$

Taking the (19),(20)and (21), we get

$$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{288}|\mathbb{S}_1^3 + 9\mathbb{S}_3\mathbb{S}_2 - 12\mathbb{S}_3| \quad .$$

By taking Lemma (2.1) with $\mathbb{S}_3 = \mathbb{S} \in [0,1]$, to get

$$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{288}|4\mathbb{S}_1^3 + 3(4 - \mathbb{S}_1^2)\mathbb{s}_1\mathcal{X}^2 - 6(4 - \mathbb{S}_1^2)(1 - |\mathcal{X}|^2)\xi| .$$

Now, by using $|\xi| \leq 1$ *and* $|\mathcal{X}| = d \leq 1$ and taking the tringle inequality,we have

$$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{288}|4\mathbb{S}^3 + 3(4 - \mathbb{S}^2)(\mathbb{S} - 2)d^2 - 6(4 - \mathbb{S}^2)| = \pi(\mathbb{S}, d).$$

Now, differentiale $\pi(\mathbb{S}, d)$ with respect to d and taking $\pi'(\mathbb{S}, d) \leq 0$ on the $[0,2] \times [0,1]$.

Put $d = 0$ to get

$\max\{\pi(\mathbb{S}, d)\} = \pi(\mathbb{S}, 0)$ .to find

$$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{288}|4\mathbb{S}^3 - 6(4 - \mathbb{S}^2)| = \mathfrak{S}(\mathbb{S}).$$

Taking $\mathfrak{S}(\mathbb{S}) = 0,$ *to get* $\mathbb{S} = 0,1$ therefore $\mathfrak{S}(\mathbb{S})$ has its maximum value at $\mathbb{S} = 0$,also that

$$|\mathfrak{a}_2\mathfrak{a}_3 - \mathfrak{a}_4| \leq \frac{1}{288}(24) = \frac{1}{12} \quad ,$$

In which the equality hold ,true for the extremal function as followes:

$$f_3 = ze^{\left(\int_0^z \frac{(1+tanht^3)-1}{t} dt\right)}$$

$$= z + \frac{z^4}{3} + \frac{z^7}{18} - \frac{z^{10}}{162} + \cdots.$$

Theorem (3.3): Suppose that the function $f \in \mathfrak{D}_\varrho^\varepsilon$ and be taking by (1.1), then, we have

$$|\mathbb{H}\mathfrak{D}_{2,2}(f)| \leq \frac{1}{18}. \quad (3.10)$$

Proof: Now, putting $\mathbb{H}\mathfrak{D}_{2,2}(f)$ by formula:

$$\mathbb{H}\mathfrak{D}_{2,2}(f) = |\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2|.$$

Taking the equations (3.6),(3.7)and (3.8), to get

$$|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2| = \frac{1}{1152}|-\mathbb{S}_1^4 - 3\mathbb{S}_1^2\mathbb{S}_2 - 12\mathbb{S}_1\mathbb{S}_3 - 8\mathbb{S}_2^2| .$$

By taking the two equations (3.6) and (3.7) in order to exact in terms of $\mathbb{S}_1$ such that $\mathbb{S}_1 = \mathbb{S}(0 \leq \mathbb{S} \leq 2$, to get

$$|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2| = \frac{1}{1152}\left|-\frac{7\mathbb{S}^4}{4} - 3(4-\mathbb{S}^2)\mathbb{S}^2\mathcal{X}^2 + 6\mathbb{S}(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)\xi - \frac{8(4-\mathbb{S}^2)^2\mathcal{X}^2}{4}\right|.$$

By taking the tringle inequality and let $\in [0,2]$, $|\xi| \leq 1$ *and* $|\mathcal{X}| = d \leq 1$,to get

$$|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2| = \frac{1}{1152}\left|-\frac{7\mathbb{S}^4}{4} - 3(4-\mathbb{S}^2)\mathbb{S}^2 d^2 + 6\mathbb{S}(4-\mathbb{S}^2)(1-d^2) - \frac{8(4-\mathbb{S}^2)^2 d^2}{4}\right| = \mathfrak{L}(\mathbb{S}, d).$$

From equation above, differentiating with respect to d ,to get

$$\frac{\partial\mathfrak{L}(\mathbb{S},d)}{\partial d} = \frac{1}{1152}\left(\frac{3(4-\mathbb{S}^2)(\mathbb{S}^2-8\mathbb{S}+12)d}{2}\right).$$

Now, we show that $\mathfrak{L}'(\mathbb{S}, d) \geq 0$ on $[0,1]$,also $\mathfrak{L}(\mathbb{S}, d) \leq \mathfrak{L}(\mathbb{S}, 1)$.

Suppose $d = 1$, to obtain

$$|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2| = \frac{1}{1152}\left[-\frac{7\mathbb{S}^4}{4} - 3(4-\mathbb{S}^2)\mathbb{S}^2 - 4(4-\mathbb{S}^2)^2\right] = \mathfrak{S}(\mathbb{S}).$$

Taking $\mathfrak{S}(\mathbb{S}) \leq 0$, where the function $\mathfrak{S}(\mathbb{S})$ be a decreasing at point $\mathbb{S}$,also $\mathfrak{S}(\mathbb{S})$ has its maximum value at $\mathbb{S} = 0$,such that:

$$|\mathbb{H}\mathfrak{D}_{2,2}(f)| \leq \frac{1}{18} \quad ,$$

In which the equality hold ,true for the extremal function as:

$$f_2 = ze^{\left(\int_0^z \frac{(1+tanht^3)-1}{t} dt\right)}$$

$$= z + \frac{z^3}{2} + \frac{z^5}{8} - \frac{5z^7}{1152} + \cdots. \quad (3.11)$$

Theorem (3.4): Suppose that the function $f \in \mathfrak{D}_\varrho^\xi$ and be taking by (1.1), then, we obtain

$$|\mathbb{H}\mathfrak{D}_{3,1}(f)| \leq \frac{1}{144} \quad . \quad (3.12)$$

Proof: We can write the third-order Hankel determinants as follows:

$$\mathbb{H}\mathfrak{D}_{3,1}(f) = 2\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4 - \mathfrak{a}_2^2\mathfrak{a}_5 - \mathfrak{a}_3^3 + \mathfrak{a}_3\mathfrak{a}_5 - \mathfrak{a}_4^2.$$

Let $\mathbb{S}_1 = \mathbb{S} \in [0,2]$ and applying (3.6), (3.7), (3.8), and (3.9) , to get

$$\mathbb{H}\mathfrak{D}_{3,1}(f) = \frac{1}{829440}[144\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3 + 276\mathbb{S}_1^4\mathbb{S}_2 - 414\mathbb{S}_1^2\mathbb{S}_2^2 - 1296\mathbb{S}_1^2\mathbb{S}_4 +$$

$$192\mathbb{S}_1^3\mathbb{S}_3 - 80\mathbb{S}_1^6 - 912\mathbb{S}_2^3 + 1728\mathbb{S}_2\mathbb{S}_4 + 1440\mathbb{S}_3^2] . \quad (3.13)$$

Let $t = 4 - \mathbb{S}^2$, appling Lemma (2.1) and some computation. Then by taking simplified from of this formulas ,to get

$$276\mathbb{S}_1^4\mathbb{S}_2 = 138(\mathbb{S}^6 + \mathbb{S}^4 t\mathcal{X}),$$

$$192\mathbb{S}^3\mathbb{S}_3 = 48\mathbb{S}^6 + 96\mathbb{S}^4 t\mathcal{X} - 48\mathbb{S}^4 t\mathcal{X}^2 + 96\mathbb{S}^3 t(1 - |\mathcal{X}|^2)\xi \ ,$$

$$414\mathbb{S}^2\mathbb{S}_2^2 = \frac{207}{2}(\mathbb{S}^6 + 2\mathbb{S}^4 t\mathcal{X} + \mathbb{S}^2 t^2\mathcal{X}^2) \ ,$$

$$1292\mathbb{S}_1^2\mathbb{S}_4 = 162\mathbb{S}^4 t\mathcal{X}^3 - 648\,\mathbb{S}^2 t\overline{\mathcal{X}}(1 - |\mathcal{X}|)\xi^2 - 648\mathbb{S}^3\mathcal{X}t(1 - |\mathcal{X}|^2)\xi$$

$$-486\mathbb{S}^3 t(1 - |\mathcal{X}|^2)\xi + 486\mathbb{S}^4 t\mathcal{X} + 162\mathbb{S}^6 + 648\mathbb{S}^2 t\mathcal{X}^2 \ ,$$

$$144\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3 = -18\mathbb{S}^2 t\mathcal{X}^2 - 18\mathbb{S}^4 t\mathcal{X}^2 + 51\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi +$$

$$37\mathbb{S}^2 t^2\mathcal{X}^2 + 37\mathbb{S}^3 t(1 - |\mathcal{X}|^2)\xi + 18\mathbb{S}^6 + 54\mathbb{S}^4 t\mathcal{X} \ ,$$

$$912\mathbb{S}_2^2 = 114(t^2\mathcal{X}^3 + 6\mathbb{S}^2 t^2\mathcal{X}^2 + 62\mathbb{S}^4 t\mathcal{X} + \mathbb{S}^6),$$

$$1728\mathbb{S}_2\mathbb{S}_4 = 108(11\mathbb{S}^2 t\mathcal{X}^2 + 11t^2\mathcal{X}^3 + \mathbb{S}^6 + 11\mathbb{S}^4 t\mathcal{X} + 11\mathbb{S}^3 t(1 - |\mathcal{X}|^2)\xi +$$

$$11\,\mathbb{S}^3 t(1 - |\mathcal{X}|^2)(1 - |\xi|^2)\zeta + 8\mathbb{S}^2 t^2\mathcal{X}^2 + 11\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi +$$

$$11t^2\mathcal{X}^3(1 - |\mathcal{X}|^2)\,(1 - |\xi|^2)\zeta - 8\mathbb{S}^4 t\mathcal{X}^2 - 11\mathbb{S}^2 t\mathcal{X}(1 - |\mathcal{X}|^2)\xi 8\mathbb{S}^2 t^2\mathcal{X}^3 \ - 11\,\mathbb{S}^2 t\overline{\mathcal{X}}(1 - |\mathcal{X}|)\xi^2$$
$$- 11\mathbb{S}t^2\mathcal{X}^2\,(1 - |\mathcal{X}|^2)\xi -$$

$$11\mathcal{X}t^2\overline{\mathcal{X}}(1 - |\mathcal{X}|)\xi^2 + \mathbb{S}^4 t\mathcal{X}^3 - \mathbb{S}^2 t^2\mathcal{X}^2),$$

and

$$1440\mathbb{S}_3^2 = 90\mathbb{S}^2 t^2\mathcal{X}^4 - 360\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi - 360\mathbb{S}^2 t^2\mathcal{X}^3 - 180\mathbb{S}^4 t\mathcal{X}^2 +$$

$$3\,60t^2(1 - |\mathcal{X}|^2)^2\xi^2 + \ 720\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi + 360\mathbb{S}^3 t(1 - |\mathcal{X}|^2)\xi +$$

$$360\mathbb{S}^2 t^2\mathcal{X}^2 + \ 360\mathbb{S}^4 t\mathcal{X} + 90\mathbb{S}^6 \ .$$

Now,put the equations above into (3.13) and simplifying,to get

$$\mathbb{H}\mathfrak{D}_{3,1}(f) = \frac{1}{829440}\Bigg[-\frac{907\mathbb{S}^6}{2} + 313\,\mathbb{S}^3 t\,(1 - |\mathcal{X}|^2)\xi + 1188t\mathcal{X}^2 - 540\mathbb{S}t^2(1 - |\mathcal{X}|^2)(1 - |\xi|^2)\zeta$$
$$+ 519\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi - 114\mathbb{S}^3\mathcal{X}^3 + 1188t^2\mathcal{X}^3(1 - |\mathcal{X}|^2)\,(1 - |\xi|^2)\zeta$$
$$+ 540\mathbb{S}^3 t\mathcal{X}(1 - |\mathcal{X}|^2)\xi + 540\,\mathbb{S}^2 t\overline{\mathcal{X}}(1 - |\mathcal{X}|)\xi^2 - 828\,\mathbb{S}t^2\mathcal{X}(1 - |\mathcal{X}|^2)\xi$$
$$- 1188\mathcal{X}t^2\overline{\mathcal{X}}(1 - |\mathcal{X}|)\xi^2 - 360t^2(1 - |\mathcal{X}|^2)^2\xi^2 - 264\mathbb{S}^4 t\mathcal{X}^2 - 853\mathbb{S}^2 t^2\mathcal{X}^2 - 522\mathbb{S}^2 t^2\mathcal{X}^3$$

$$-1188\mathbb{S}^2 t\mathcal{X}^2 + 90\mathbb{S}^2 t^2\mathcal{X}^4 - 54\mathbb{S}^4 t\mathcal{X}^3 + 34\mathbb{S}^4 t\mathcal{X}\ \Bigg].$$

Let, $t = (4 - \mathbb{S}^2)$,to get

$$\mathbb{H}\mathfrak{D}_{3,1}(f) = \frac{1}{829440}[\sigma_1(\mathbb{S},\mathcal{X}) + \sigma_2(\mathbb{S},\mathcal{X}) + \sigma_3(\mathbb{S},\mathcal{X}) + \hbar(\mathbb{S},\mathcal{X},\xi)\zeta],$$

Such that

$\sigma_1(\mathbb{S},\mathcal{X}) = -2\mathcal{X}(4-\mathbb{S}^2)[3\mathcal{X}(4-\mathbb{S}^2)(-15\mathbb{S}^2\mathcal{X}^2 + 87\mathbb{S}\,\mathcal{X}^2 + 142\mathbb{S}^2 + 78\mathcal{X}\,) + 27\mathbb{S}^4\mathcal{X}^2 - 264\mathbb{S}^4\mathcal{X} + 17\mathbb{S}^4 + 852\mathbb{S}\,\mathcal{X}^2] + \frac{907\mathbb{S}^6}{2}$,

$\sigma_2(\mathbb{S},\mathcal{X}) = -8(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)\mathbb{S}[(207\mathcal{X}^2 - 520\mathcal{X})(4-\mathbb{S}^2) - 135\mathbb{S}^4\mathcal{X} - 39\mathbb{S}^2\,]$,

$\sigma_3(\mathbb{S},\mathcal{X}) = -4(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)[(297\mathcal{X}^2 + 90)\,(4-\mathbb{S}^2) - 135\mathbb{S}^2\bar{\mathcal{X}}]$, and

$\hbar_4(\mathbb{S},\mathcal{X},\xi) = 4(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)(1-|\xi|^2)[543(4-\mathbb{S}^2)\mathcal{X} - 135\,\mathbb{S}^2]$.

Let $\zeta \leq 1$, $|\mathcal{X}| = \mathcal{X} and\ |\xi| = \ell$, to get

$\mathbb{H}\mathfrak{D}_{3,1}(f) = \frac{1}{829440}\left[|\sigma_1(\mathbb{S},\mathcal{X})| + |\sigma_2(\mathbb{S},\mathcal{X})|\ell + |\sigma_3(\mathbb{S},\mathcal{X})|\ell^2 + |\hbar(\mathbb{S},\mathcal{X},\xi)|\right]$,

$$\leq \frac{1}{829440}[\Upsilon(\mathbb{S},\mathcal{X},\ell)]\ , \tag{3.14}$$

Such that

$$\Upsilon(\mathbb{S},\mathcal{X},\ell) = \left[\ell_1(\mathbb{S},\mathcal{X}) + \ell_2(\mathbb{S},\mathcal{X})\ell + \ell_3(\mathbb{S},\mathcal{X})\ell^2 + \ell_4(\mathbb{S},\mathcal{X})\left(1-\ell^2\right)\right], \tag{3.15}$$

where

$\ell_1(\mathbb{S},\mathcal{X}) = 2\mathcal{X}(4-\mathbb{S}^2)[3\mathcal{X}(4-\mathbb{S}^2)(-15\mathbb{S}^2\mathcal{X}^2 + 87\mathbb{S}\,\mathcal{X}^2 + 142\mathbb{S}^2 + 178\mathcal{X}\,) + 27\mathbb{S}^4\mathcal{X}^2 + 132\mathbb{S}^4\mathcal{X} - 17\mathbb{S}^4 + 594\mathbb{S}\,\mathcal{X}^2] - \frac{907\mathbb{S}^6}{2}$,

$\ell_2(\mathbb{S},\mathcal{X}) = 8(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)\mathbb{S}[(-207\mathcal{X}^2 - 65\mathcal{X})(4-\mathbb{S}^2) + 135\mathbb{S}^4\mathcal{X} + 39\mathbb{S}^2\,]$,

$\ell_4(\mathbb{S},\mathcal{X}) = 4(4-\mathbb{S}^2)(1-|\mathcal{X}|^2)(1-|\xi|^2)[543(4-\mathbb{S}^2)\mathcal{X} - 135\,\mathbb{S}^2]$,

Suppose the closed cuboid be as follows:

$\mathfrak{y}: [0,2] \times [0,1], [0,1]$.

To find the insid maxima points ,where contains twelve edges and six faces to maximize the function $\Upsilon(\mathbb{S},\mathcal{X},\ell)$ which defined by (3.15).

Now, we put three cases.

1.Suppose that $\mathbb{S},\mathcal{X},\ell \in (0,2)\times(0,1)\times(0,1)$ .By taking partial derivative of (3.15) with respect to ℓ , to found the points of maxima inside $\mathfrak{y}$.

$$\frac{\partial\Upsilon}{\partial\ell} = 4\left[36(1-\mathcal{X})[(\mathcal{X}-5)(4-\mathbb{S}^2) + 135\mathbb{S}^2\,] + \mathbb{S}\left(360\mathcal{X}(4-\mathbb{S}^2)(13-\mathcal{X}) + \mathbb{S}^2\,(540\mathcal{X}+313)\right)\right], \tag{3.16}$$

now, we can find

$\ell = \frac{\mathbb{S}[360\mathcal{X}(4-\mathbb{S}^2)(\mathcal{X}-13)-\mathbb{S}^2(540\mathcal{X}-313)]}{36(\mathcal{X}-1)[(\mathcal{X}-5)(4-\mathbb{S}^2)+135\,\mathbb{S}^2]}$.

Let ℓ_0 be a critical point inside $\mathfrak{y}$,thus $\ell_0 \in (0,1)$, which is possible if

$\mathbb{S}[360\mathcal{X}(4-\mathbb{S}^2)(\mathcal{X}-13)-\mathbb{S}^2(540\mathcal{X}-313)]-36(\mathcal{X}-1)[(1-\mathcal{X})(4-\mathbb{S}^2)(5-\mathcal{X})\,] <$
$-4860(1-\mathcal{X})\mathbb{S}^2$. (3.17)

And

$\mathbb{S}^2 > \frac{24(5-\mathcal{X})}{1530-\mathcal{X}}$. (3,18)

To obtain the solutions in order to satisfying two both of inequalities (3.17) and (3.18).

Suppose that ,

$\mathscr{b}(\mathcal{X}) = \frac{24(5-\mathcal{X})}{1530-\mathcal{X}}$. (3.19)

In the interval (0,1) the function $\mathscr{b}(\mathcal{X})$ be decreasing where $\mathscr{b}'(\mathcal{X}) < 0$ on (0,1). There is no critical point of $\Upsilon(\mathbb{S},\mathcal{X},\ell)$ in$(0,2)\times(0,1)\times(0,1)$. Where (3.17) does not hold true in this case for all values of $\mathcal{X} \in (0,1)$.

2. To find inside maxima points of the six faces of the cuboid $\mathfrak{y}$. .We put $\mathbb{S} = 0, \Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\mathcal{T}_1(\mathcal{X},\ell) =$

$-1900\mathcal{X}^3 + 4(1-\mathcal{X}^2)(4752\mathcal{X}^2+1440)\ell^2 + 19008\mathcal{X}(1-\mathcal{X}^2)(1-\ell^2), \quad (\mathcal{X},\ell \in (0,1))$.
(3.20)

Now, in (0,1) × (0,1) the function $\mathcal{T}_1(\mathcal{X},\ell)$ has no optimal points since

$\frac{\partial \mathcal{T}_1(\mathcal{X},\ell)}{\partial \ell} = 8(1-\mathcal{X}^2)(4752\mathcal{X}^2+1440)\ell + 38016\mathcal{X}\ell(1-\mathcal{X}^2), (\mathcal{X},\ell \in (0,1))$.

(3.21)

Put $\mathbb{S} = 2, \Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\Upsilon(2,\mathcal{X},\ell) = 29024$, $(\mathcal{X},\ell \in (0,1))$. (3.22)

Put $\mathcal{X} = 2, \Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\mathcal{T}_2(\mathbb{S},\ell) = \frac{907\mathbb{S}^6}{2} + 313(4-\mathbb{S}^2)\mathbb{S}\ell + (4-\mathbb{S}^2)(1440+360\mathbb{S}^2)\ell^2 + \mathbb{S}^2\,(4-\mathbb{S}^2)\left(1-\ell^2\right)$.

(3.23)

Now, we solve (3.23) such that $\ell \in (0,1)$ *and* $\mathbb{S} \in (0,2)$.

$\frac{\partial \mathcal{T}_2(\mathbb{S},\ell)}{\partial \ell} = 0$ and $\frac{\partial \mathcal{T}_2(\mathbb{S},\ell)}{\partial \mathbb{S}} = 0$, on solving $\frac{\partial \mathcal{T}_2(\mathbb{S},\ell)}{\partial \ell} = 0$ to obtain the points of maxima.

We have

$\ell = \frac{313\mathbb{S}^2}{2(1440\mathbb{S}^2-359)} =: \ell_1$. (3.24)

In (0,1) we have the range of ℓ, ℓ_1 , where

$\mathbb{S} > \mathbb{S}_0$, $\mathbb{S}_0 \cong 1.54572016538129$.

A computation shows that

$$\frac{\partial \mathcal{T}_2(\mathbb{S},\ell)}{\partial \mathbb{S}} = 0$$

Hence,we have

$2721\mathbb{S}^2 - 1565\mathbb{S}^4\ell + 3756\mathbb{S}^2\ell + 1436\mathbb{S}^3\ell^2 - 8\mathbb{S}\ell^2 - 4\mathbb{S}^3 + 8\mathbb{S} = 0$. (3.25)

Put equation (3.24) into equation (3.25) to get

$\mathbb{S}(2686.983\mathbb{S}^4 - 170.085\mathbb{S}^3 + 682.235\mathbb{S}^5 + 402.973\mathbb{S} + 17.15\mathbb{S}^2 + 272.844\mathbb{S}^6 - 0.094) = 0.$

(3.26)

We give the solution of equation (3.26) in (0,2) , that is ,

$\mathbb{S} \cong 1.1776484167107$. Therefore in$(0,2) \times (0,1)$ the equation $\mathcal{T}_2$ has no optimal point.

Put $\mathcal{X} = 1, \Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\mathcal{T}_3(\mathbb{S},\ell) = 3374\mathbb{S}^6 - 11600\mathbb{S}^4 + 2976\mathbb{S}^2 + 17088$. (3.27)

Solving the equation

$\frac{\partial \mathcal{T}_3(\mathbb{S},\ell)}{\partial \mathbb{S}} = 0$.

To get the critical points as follows

$\mathbb{S} = \mathbb{S}_0 = 0$, $\mathbb{S}_1 \cong 0.369307722$.

Because $\mathbb{S}_0$ be the minimum point of $\mathcal{T}_3(\mathbb{S},\ell)$, $\mathcal{T}_3(\mathbb{S},\ell)$ attains it maximum value at $\mathbb{S}_1$, that is , $\mathbb{S} = 17286.67112$.

Put $\ell = 0$, $\Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\mathcal{T}_4(\mathbb{S},\mathcal{X}) = \Upsilon(\mathbb{S},\mathcal{X},0) =$

$\frac{907\mathbb{S}^6}{2} + (8-2\mathbb{S}^2)\mathcal{X}[(8-2\mathbb{S}^2)\mathcal{X}(87\mathbb{S}^2\mathcal{X} - 15\mathbb{S}^2\mathcal{X}^2 + 142\mathbb{S}^2 - 178\mathcal{X}) + 27\mathbb{S}^4\mathcal{X}^2 - 132\mathbb{S}^4\mathcal{X} + 17\mathbb{S}^4 - 594\mathbb{S}^2\mathcal{X}]$

$+(8-2\mathbb{S}^2)(1-\mathcal{X}^2)[297(4-\mathbb{S}^2)\mathcal{X} + 135\mathbb{S}^2]$.

The calculations shows that no solution of the following equations:

$\frac{\partial \mathcal{T}_4(\mathbb{S},\mathcal{X})}{\partial \mathcal{X}} = 0$ and $\frac{\partial \mathcal{T}_4(\mathbb{S},\mathcal{X})}{\partial \mathbb{S}} = 0$, in$(0,2) \times (0,1)$.

Put $\ell = 1$, $\Upsilon(\mathbb{S},\mathcal{X},\ell)$,we have

$\mathcal{T}_5(\mathbb{S},\mathcal{X}) = \Upsilon(\mathbb{S},\mathcal{X},1) =$

$\frac{907\mathbb{S}^6}{2} + (8-2\mathbb{S}^2)\mathcal{X}[(8-2\mathbb{S}^2)\mathcal{X}(87\mathbb{S}^2\mathcal{X} - 15\mathbb{S}^2\mathcal{X}^2 + 142\mathbb{S}^2 - 178\mathcal{X}) + 27\mathbb{S}^4\mathcal{X}^2 - 132\mathbb{S}^4\mathcal{X} + 17\mathbb{S}^4 - 594\mathbb{S}^2\mathcal{X}]$

$+(32-8\mathbb{S}^2)(1-\mathcal{X}^2)\mathbb{S}[(65\mathcal{X} - 207\mathcal{X}^2)(4-\mathbb{S}^2) + 135\mathbb{S}^2\mathcal{X} + 39\mathbb{S}^2]$

$+ (32-8\mathbb{S}^2)(1-\mathcal{X}^2)[(297\mathcal{X}^2 + 90)(4-\mathbb{S}^2) + 135\mathbb{S}^2\mathcal{X}]$.

The calculations shows that no solution of the following equations:

$\frac{\partial \mathcal{T}_5(\mathbb{S},\mathcal{X})}{\partial \mathcal{X}} = 0$ and $\frac{\partial \mathcal{T}_5(\mathbb{S},\mathcal{X})}{\partial \mathbb{S}} = 0$, in$(0,2) \times (0,1)$.

3. To found the maxima of $\Upsilon(\mathbb{S},\mathcal{X},\ell)$on the edges of η. Put $\ell = 0$ in, (3.23) we have

$\Upsilon(\mathbb{S},0,0) = \eta_1(\mathbb{S}) = \frac{907\mathbb{S}^6}{2} - 1840\mathbb{S}^4 + 1660\mathbb{S}^2$.

Now $\eta_1'(\mathbb{S}) = 0$ for $\mathbb{S} = \eta_0 = 0$ and $\mathbb{S} = \eta_1 = 1.86233979316127$ in $[0,2]$,such that $\eta_0$be minimum point and the maximum point of $\eta_1(\mathbb{S})$ is attained at $\eta_1(\mathbb{S})$, we have,

$\Upsilon(\mathbb{S},0,0) \leq 216.2934888$,$\mathbb{S} \in [0,1]$.

From (3.23) where $\ell = 1$, we have

$\Upsilon(\mathbb{S},0,0) = \eta_2(\mathbb{S}) = \frac{907\mathbb{S}^6}{2} - 1080\mathbb{S}^5 + 360\mathbb{S}^4 + 4320\mathbb{S}^3 - 2880\mathbb{S}^2 + 5760$.

,be decreasing in $[0,2]$, where ,$\eta_2'(\mathbb{S}) < 0$, $\mathbb{S} \in [0,2]$ and hence the Now, $\eta_2(\mathbb{S})$

Maximum is obtained at $= 0$. therefore

$\Upsilon(\mathbb{S},0,1) \leq 5760$,$\mathbb{S} \in [0,1]$.

Let $\mathbb{S} = 0$ in equation (3.23) , to obtain

$\Upsilon(0,0,\ell) \leq 5760\ell^2$, $\mathbb{S} \in [0,1]$.

A simple calculation gives

$\Upsilon(0,0,\ell) \leq 5760$, $\ell \in [0,1]$.

From (3.27),we get

$\Upsilon(\mathbb{S},1,1) = \Upsilon(\mathbb{S},1,0) = \eta_3(\mathbb{S}) = 3374\mathbb{S}^6 - 11600\ \mathbb{S}^4 + 2476\mathbb{S}^2 + 17088$.

Now, $\mathbb{S} = \eta_1 = 0.334548402\ in\ [0,2]$and $\eta_3'(\mathbb{S}) = 0$ for $\mathbb{S} = \eta_0 = 0$ such that η_0 be the minimum point and the maximum point of $\eta_3(\mathbb{S})$ is satisfied by at $\eta_1(\mathbb{S})$. Lead to the following equation.

$\Upsilon(\mathbb{S},1,1) = \Upsilon(\mathbb{S},1,0) \leq 17224.54141\ , \mathbb{S} \in [0,2]$.

From equation (3.27) , where $\mathbb{S} = 0$ to get

$\Upsilon(0,1,\ell) = 17088$.

From equation (3.22) ,we obtain

$\Upsilon(2,0,\ell) = \Upsilon(2,1,\ell) = \Upsilon(2,\mathcal{X},0) = \Upsilon(2,\mathcal{X},1) = 29024 \quad ,(\mathcal{X},\ell \in [0,1])$.

Let $\ell = 0$ in (3.20),to get

$\Upsilon(0,\mathcal{X},0) = \eta_4(\mathbb{S}) = -17088\mathcal{X}^3 + 34176\,\mathcal{X}$.

Now, $\mathcal{X} = \mathcal{X}_0 = 0.81649658\ in\ [0,1]$ where $\eta_4'(\mathbb{S}) = 0$ such that η_4 be increasing for $\mathcal{X} \leq \mathcal{X}_0$ and decreasing for$\mathcal{X}_0 \leq \mathcal{X}$. Thus η_4 has its maximum point at $\mathcal{X} = \mathcal{X}_0$. This implies that.

$\Upsilon(0,\mathcal{X},0) \leq 13952.29356\ , \quad (\mathcal{X} \in [0,1])$.

From equation (3.20) and let $\ell = 1$, to get

$\Upsilon(0,\mathcal{X},1) = \eta_5(\mathcal{X}) = 17088\mathcal{X}^3 - 3860\mathcal{X}^2 + 5760 - 19008\mathcal{X}^4$.

Now, $\eta_5(\mathcal{X})$ is decreasing in $[0,1]$ where$\eta_5'(\mathcal{X}) < 0\ for\ [0,1]$ and then at lest maximum value at $\mathcal{X} = 0$,also

$\Upsilon(0,\mathcal{X},1) =\leq 5760\ ,(\mathcal{X} \in [0,1])$.

Therefore , from the above cases, we have

$\Upsilon(\mathbb{S},\mathcal{X},\ell) \leq 5760 \quad ,on\ [0,2] \times [0,1] \times [0,1]$. (3.28)

From the two equations (3.14) and (3.28) we obtain

$\mathbb{H}\mathfrak{D}_{3,1}(f) = \frac{1}{829440}[\Upsilon(\mathbb{S},\mathcal{X},\ell)] \leq \frac{1}{144}$.

If $f \in \mathfrak{D}_{\varrho}^{\varepsilon}$,thus the equality is carried out by the function represent by

$$f_2 = ze^{\left(\int_0^z \frac{(1+tanht^3)-1}{t} dt\right)}$$

$= z + \frac{z^4}{12} + \frac{z^7}{24} - \frac{5z^{10}}{3456} + \cdots$. (3.29)

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