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STABILITY,ERROR ESTIMATE AND APPROXIMATION OF SOLUTIONS OF NON-LINEAR SYSTEM INTEGRAL EQUATIONS IN DIRECT SUM OF HILBERT SPACE

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Abstract

In this research, we study stability of solutions with respect to the driver of a system of non-linear Volterrs integral equations in reproducing Kernel Hilbert space. We study error estimate of solutions in terms of uniform grid numbers of subintervals of [a, b]. Furthermore, we use the reproducing kernel method to approximate the solutions of the problem. Finally, we give some examples to show the power of the method.

Keywords: Reproducing Kernel Hilbert Space, System of Non-Linear Integral Equation, Stability of solutions.

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Introduction

Non-linear integral equations has been used in numerous fields of science and engineering, including elasticity, mass transfer, oscillation theory, fluid dynamics, biomechanics, game theory, control theory, electrical engineering, and economics. Most of the time, there are different kinds of non-linear integral equations that cannot be worked out exactly, so it is best to approach them in an approximate way [1-9]. Therefore, many researchers studied and focused on different numerical techniques which can work out these integral equations. For instance, the authors presented the homotopy analysis method to solve the second kind of non-linear Fredholm and Volterra integral equations [1, 2], collocation method by [10], and singular integral equations and other numerical techniques presented by [3-5]. Existence and uniqueness of solutions of systems of non-linear integral equations in direct sum of reproducing kernel Hilbert space is presented by [6].

The goal of the paper is to present an approximate representation for solving systems of general class of non-linear Volterra integral equations in a direct sum of reproducing kernel Hilbert space H[a, b]; see Definition 2.4. The stability of the proposed method is proved and the error estimate is investigated in this work as well. We consider for all $t \in [a-b]$:

$$f(t) = \alpha(t) + \int_a^t F\big(t, s, f(s), g(s)\big)ds,$$

$$g(t) = \beta(t) + \int_a^t G\big(t, s, f(s), g(s)\big)ds, \qquad (1,1)$$

where $\alpha, \beta: R \to V[a,b]$ are given functions. The unknown functions f, g need to be determined, and F, G satisfy the given regularity conditions; see [6].

We organized this paper into six sections including the introduction In section 2, we present basic definitions and related notations for reproducing kernel Hilbert space. We introduce the representation of solutions for the proposed problem in Section 3. In section 4, we study stability and error estimate for the solutions in H[a,b]. We give the numerical experiments in Section 5. Finally, in section 6 ends this paper with a conclusion.

2. BASIC DEFINITIONS

Definition 2.1. [5] Let $H[a,b]$ be a function Hilbert space, including all real value functions $h: X \to R$ where X is a on a nonempty abstract set, with the inner product $\langle\cdot,\cdot\rangle_{H[a,b]}$. For all fixed $x \in X$ and if there exists a function $k_x(.) \in H$ such that

$$\langle h, k_x\rangle_{H[a,b]} = h(x)$$

for all $x \in X$ and all $h \in H$. Such a function $k = k_x(.)$ is called the reproducing kernel function of $H[a.b]$ and the Hilbert space $H[a,b]$ is called the reproducing kernel Hilbert space.

Definition 2.2. [11] Let V$[a,b]$ be the space of all absolutely continuous functions $f : [a,b] \to R$ such that $f' \epsilon L^2[a,b]$.

Theorem 2.3. [9] The function space V[a, b] equipped with the inner product

$$\langle f_1 , f_2 \rangle_{V[a,b]} = f_1(a)f_2(a) + \int_a^b f_1'(t)f_2'(t)\ dt,$$

and associated with the norm.

$\|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle_{V[a,b]}},$

is a reproducing kernel Hilbert space and the reproducing kernl function

$$K = K(\cdot,\cdot) \text{ is defined by:}$$

$$K(s,t) = \begin{cases} t+1-a & if \quad a \le t \le s \le b, \\ s+1-a & if \quad a \le s \le t \le b, \end{cases}$$

Definition 2.4. The function space

$$H[a,b] = V[a,b] \oplus V[a,b],$$

Consists of those functions $\vec{h}: [a,b] \to R^2$ *where* $\vec{h} = (h_1, h_2)$*such that* h_1 *and* h_2 *belong to* $V[a, b]$.

Definition 2.5. The inner product of the space $H[a,b]$ is defined by:

$\langle \vec{f}, \vec{g} \rangle_{H[a,b]} = \langle f_1, g_1 \rangle_{V[a,b]} + \langle f_2, g_2 \rangle_{V[a,b]}$,

where $f = (f_2, f_2)$ *and* $g = (g_1, g_2)$. Such a space is called direct sum of the reproducing kernel Hilbert space $V[a,b]$.

3. APPROXIMATION OF SOLUTION

Let $\langle H[a,b], \langle \cdot,\cdot \rangle_{H[a,b]} \rangle$ be reproducing kernel Hilbert space of continuous functions on a set W in R, with reproducing kernel functions k, and let $T: H[a,b] \to H[a,b]$ be a one-to-one, bounded, linear transformation. If $\vec{h} \in$ H$[a,b]$ is a solution to

$$T\vec{h} = \vec{h}$$

for a given $\vec{h} \in H\ [a,b]$, then $\vec{h}$ may be expressed in terms of a complete orthonormal basis for $H[a,b]$ generated using T. For more details and proofs of the results in this section [9].

Let $\{u_i\}_{i=1}^{\infty}$ be a countable set of distinct points in W, and define

$$R_i = T^* k_{u_i},$$

where $T^* : H[a,b] \to H[a,b]$ is the adjoint operator of T.

Theorem 3.1. If $\{u_i\}_{i=1}^{\infty}$ is dense in W. Then $\{R\}_{i=1}^{\infty}$ is a complete set in H[a, b] and

$$R_i = Tk_{u_i} \qquad \forall_i \in N$$

Can then be derived by applying the Gram-Schmidt orthonormalization process to $\{R\}_{i=1}^{\infty}$:

$$R_K = \sum_{k=1}^{i} \beta_{ik} R_{k,}$$

where the β_{ik} are orthonormalization coefficients of $\{R_i\}_{i=1}^{\infty}$

Theorem 3.2. Let $\{u_i\}_{i=1}^{\infty}$ be a countable dense set of points ofW , let $\vec{h} \in H[a,b]$be solution of $T\vec{h} = \vec{h}$. Then $\vec{h}$ has the following Hilbert space representation:

$$\vec{h} = \sum_{i=1}^{\infty} \sum_{i=1}^{i} \beta_{ik} \vec{h}(uk) R_{K.}$$

We observe that the truncation

$$\vec{h}_n = \sum_{i=1}^{n} \sum_{i=1}^{i} \beta_{ik} \vec{h}(uk) R_{K.} \qquad (3,1)$$

is an approximation of the exact solution $\vec{h}$ to $T\vec{h} = \vec{h}$.

4. STABILITY OF SOLUTIONS AND ERROR ESTIMATE

Let T,L denote the integral operators defined as:

$$T\vec{h}(t) = \int_a^t F(t,s,\vec{h}(s))\, ds,$$

$$L\vec{h}(t) = \int_a^t G(t,s,\vec{h}(s))\, ds,$$

Consider the nonlinear Volterra integral equation

$$T\vec{h} = \vec{h} \ \ in\ [a,b]$$

Set $\alpha, \beta \in H[a,b]$. Define operators $\Gamma : H[a,b] \to H\ [a,b]$ and $\Lambda : H[a,b] \to H[a,b]$ such that:

$$\Gamma\vec{h}(t) = \alpha(t) + T\vec{h}(t);$$

$$\Lambda\vec{h}(t) = \beta(t) + L\vec{h}(t).$$

for all $\vec{h} \in H[a,b]$. We divide the interval $[a,b]$ into $\mathbb{N}$ equally subintervals $a \le t_0 < t_1 < \cdots < t_n \le b$; where

$\Delta t = t_j - t_{j-1}$, $j = 1,2, \dots N$ and $\Delta t = \frac{b-a}{N}$. The inner product in $H[t_j, t_j + \Delta t]$ is defined by :

$$\langle \vec{f}, \vec{g} \rangle_{H[t_j,t_j+\Delta t]} = \langle f_1, g_1 \rangle_{V[t_j,t_j+\Delta t]} + \langle f_2, g_2 \rangle_{V[t_j,t_j+\Delta t]},$$

For all $\vec{f},\vec{g} \in H[t_j,t_j+\Delta t]$. As a result, we see that the operators $\Gamma: H[t_j,t_j+\Delta t] \to H[t_j,t_j+\Delta t]$ and $\Lambda: H[t_j,t_j+\Delta t] \to H[t_j,t_j+\Delta t]$

become

$$\Gamma\vec{h}(\mu) = \alpha(\mu) + \int_{t_j}^{\mu} F\left(\mu\,, s\,, \vec{h}(s)\right) ds;$$

$$\Lambda\vec{h}(\mu) = \beta(\mu) + \int_{t_j}^{\mu} G\left(\mu\,, s\,, \vec{h}(s)\right) ds.$$

for all $\mu \in H[t_j,t_j+\Delta t]$.We can rewrite above system as a matrix notation as :

$$\Pi\vec{h} = \int_{a}^{t} \Gamma\left(t\,, s\,, \vec{h}(s)\right) ds$$

where

$$\Pi = (T, L)\ and\ \Gamma = (F\,, G)$$

The stability of RKHS solutions $\vec{h}$ to $\Lambda\vec{h} = \vec{\alpha}$ with respect to the driver $\vec{\alpha}$ is examined in this section, and the approximation error is studied when the truncation $\vec{h}_n$, in (1,1) is utilise inplace of $\vec{h}$. Solutions in $H[a,b]$ will be the main emphasis of this section. The results of the RKHS existence and uniqueness tests are as follows [6].

Theorem 4.1. Let $\vec{\gamma}_1\vec{\gamma}_2$, belong to $H[t_j,t_j+\Delta t]$ and let $\vec{h}_1, \vec{h}_2$ be the unique solutions in $H[t_j,t_j+\Delta t]$ to $\Lambda\vec{h}_i = \vec{\gamma}_i$, $(i = 1,2)$. Then there exist a constant M such that:

$$\|\vec{h}_1 - \vec{h}_2\|_{H[t_j,t_j+\Delta t]} \leq M\|\vec{\gamma}_1 - \vec{\gamma}_2\|_{H[t_j,t_j+\Delta t]}.$$

Proof:

$$\|\vec{h}_1 - \vec{h}_2\|_{H[t_j,t_j+\Delta t]} = \|\vec{\gamma}_1 + \Pi\vec{h}_1 - (\vec{\gamma}_2 + \Pi\vec{h}_2)\|_{H[t_j,t_j+\Delta t]}$$

$$= \|\vec{\gamma}_1 - \vec{\gamma}_2 + \Pi\vec{h}_1 - \Pi\vec{h}_2\|_{H[tj,tj+\Delta t]}$$

$$\leq \|\vec{\gamma}_1 - \vec{\gamma}_2\|_{H[tj,tj+\Delta t]} + \|\Pi\vec{h}_1 - \Pi\vec{h}_2\|_{H[tj,tj+\Delta t]}$$

By theorem 4

$$\leq \|\vec{\gamma}_1 - \vec{\gamma}_2\|_{H[tj,tj+\Delta t]} + \delta(\Delta t)\|\vec{h}_1 - \vec{h}_2\|_{H[tj,tj+\Delta t]}$$

This implies that.

$$\|\vec{h}_1 - \vec{h}_2\|_{H[tj,tj+\Delta t]} \leq M\|\vec{\gamma}_1 - \vec{\gamma}_2\|_{H[tj,tj+\Delta t]}.$$

Where $M\frac{1}{1-\delta(\Delta t)}$ and $\delta(\Delta t) < 1$ because (Δt) is taken small enough □

.

Theorem 4.2. Let $n > 0$ and let $t_i \in [a,b]$ where $t_i = t_{i-1} + \Delta t, \Delta t = \frac{b-a}{n}$. Let $\vec{h}$ be a unique solution in H[a,b] to $A\vec{h} = \vec{h}, (A = \vec{\alpha} + \Pi\vec{h})$ and let $\vec{h}_n$, be an approximate solution of $\vec{h}$ given by (2.1) then

$$\left|\vec{h}(t) - \vec{h}_n(t)\right| \leq \frac{2}{n} \left\|A\vec{h}\right\|_{H_{[a,b]}}$$

for all $t \in [a, b]$.

Proof: For all $t \in [a, b]$ there exists $t_i \in [a, b]$ such that $|t - t_i| < \frac{1}{n}$. Since $A\vec{h}(t_i) = A\vec{h}_n(t_i)$ for all $i = 0, 1, 2, \ldots n$. We note that:

$$\begin{aligned} \left|\vec{h}(t) - \vec{h}_n(t)\right| &= \left|\vec{h}(t) - A\vec{h}(t_i) + A\vec{h}_n(t_i) - \vec{h}_n(t)\right| \\ &= \left|\vec{h}(t) - \vec{h}(t_i) + \vec{h}_n(t_i) - \vec{h}_n(t)\right| \\ &\leq \left|\vec{h}(t) - \vec{h}(t_i)\right| + \left|\vec{h}_n(t) - \vec{h}_n(t_i)\right| \end{aligned}$$

Since

$$\begin{aligned} \left|\vec{h}(t) - \vec{h}(t_i)\right| &= \left|\langle \vec{h}(\cdot), \vec{k}(\cdot, t)\rangle_{H_{[a,b]}} - \langle \vec{h}(\cdot), \vec{k}(\cdot, t_i)\rangle_{H_{[a,b]}}\right| \\ &= \left|\langle \vec{h}(\cdot), \vec{k}(\cdot, t) - \vec{k}(\cdot, t_i)\rangle_{H_{[a,b]}}\right| \\ &\leq \left\|A\vec{h}\right\|_{H_{[a,b]}} \left\|\vec{k}(.,t) - \vec{k}(.,t_i)\right\|_{H_{[a,b]}} \\ &= \left\|A\vec{h}\right\|_{H_{[a,b]}} \left\|\vec{k}_1(.,t) - \vec{k}_2(.,t_i)\right\|_{H_{[a,b]}} \end{aligned}$$

by definition (2.5)

$$\left|\vec{h} - \vec{h}(t_i)\right| \leq \left\|A\vec{h}\right\|_{H_{[a,b]}} \frac{1}{n}$$

We can use the same way to show

$$\left|\vec{h}_n - \vec{h}_n(t_i)\right| \leq \frac{1}{n} \left\|A\vec{h}\right\|_{H_{[a,b]}}$$

From equations 1 and 2 we obtain

$$\left|\vec{h}(t) - \vec{h}_n(t)\right| \leq \frac{2}{n} \left\|A\vec{h}\right\|_{H_{[a,b]}} \qquad \square$$

5 .NUMERICAL APPLICATINS

Example 5.1. Solve the system of non-linear Volterra integral equation

$$f(t) = \alpha(t) + \int_0^t \left(ts + f(s) + g(s)\right)^2 ds$$

$$g(t) = \beta(t) + \int_0^t \left(tsin(s) + f(s)g(s)\right)ds.$$

where

$$\alpha(t) = \frac{9}{2} - \frac{e^{2t}}{2} - \frac{31}{30}t^5 - 2t + (-3 + 6t - 4t^2)e^t$$

and

$$\beta(t) = 2 + tcos(t) - t + t^2 - (t^2 - 2t + 2)e^t$$

We note that an exact solution for this problem is

$$f(t) = e^t \, and \, g(t) = t^2.$$

Table 1: Numerical comparisons between the Exact solution and the Approximate solution

s	t	Exact solution	Approximate solution	Abs. Err.
0.1	0.1	1.10517	1.10517	1.13252×10^{-5}
0.2	0.2	1.2214	1.2214	5.50071×10^{-5}
0.3	0.3	1.34986	1.34979	3.50843×10^{-4}
0.4	0.4	1.49182	1.49176	9.24575×10^{-4}
0.5	0.5	1.64872	1.64866	2.01681×10^{-4}
0.6	0.6	1.82212	1.82207	3.90183×10^{-4}
0.7	0.7	2.01375	2.01297	6.94282×10^{-3}
0.8	0.8	2.22554	2.22389	1.16089×10^{-3}
0.9	0.9	2.4596	2.4581	1.84935×10^{-3}

Table 2: Numerical comparisons between the Exact solution and the Approximate solution

s	*t*	Exact solution	Approximate solution	Abs. Err.
0.1	0.1	0.01	0.01	1.0482×10^{-5}
0.2	0.2	0.04	0.04	2.18436×10^{-5}
0.3	0.3	0.09	0.09	$3.\ 39244 \times 10^{-4}$
0.4	0.4	0.16	0.16	4.65076×10^{-4}
0.5	0.5	0.25	0.25	5.93028×10^{-3}
0.6	0.6	0.36	0.36	7.19261×10^{-3}
0.7	0.7	0.49	0.49	8.38771×10^{-3}
0.8	0.8	0.64	0.64	9.45112×10^{-3}
0.9	0.9	0.81	0.81	1.03008×10^{-3}

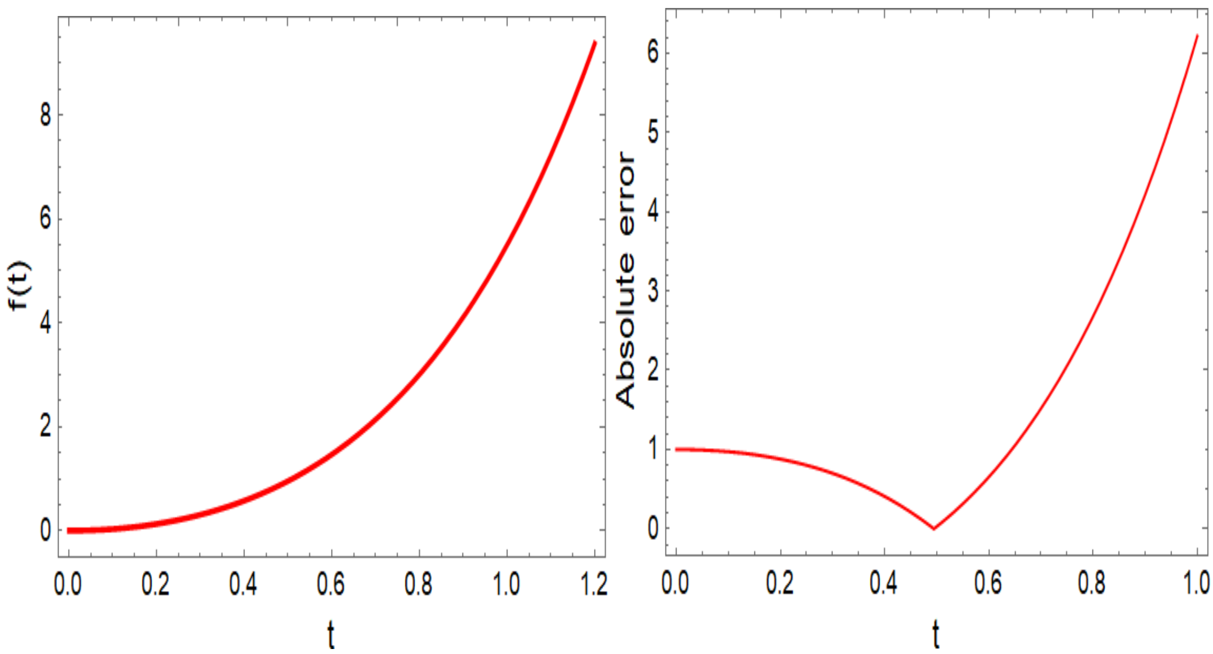

Fig.1. The proposed approach and the absolute error of $f(t)$

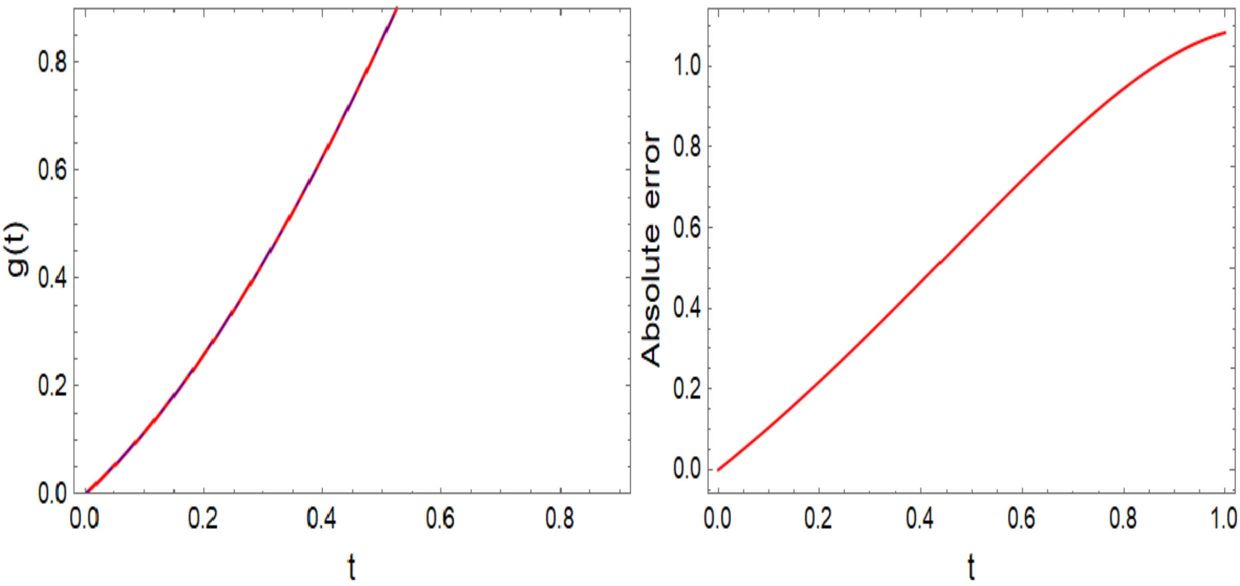

Fig.2. The proposed approach and the absolute error of $g(t)$

6. CONCLUSION

In this work, we discussed the stability of solutions of a system of non-linear integral equations on the Hilbert space $H[a,b]$. We found the upper bound error; see Theorem (4.2) of the approximate solutions which is defined in eq. (3.1). We used the reproducing kernel method to solve the problem (1.1).

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