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SOME NEW SEMI-NORMED DIFFERENCE ENTIRE SEQUENCE SPACES OF FUZZY REAL NUMBERS DEFINED BY A DOUBLE ORLICZ FUNCTIONS

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Abstract:

The purpose of this paper is to introduce semi-normed bounded, convergent and null double sequences of fuzzy real numbers defined by using a double Orlicz functions and study some properties of these sequence spaces like convergence-free, solidness , symmetric. We have proved some inclusion results too.

Keywords: Double Orlicz Function, Fuzzy Real Number, Difference Double Sequence, Semi-Normed Space.

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1. Introduction:

The concept of fuzzy set theory was introduced by Zadeh [1]. Based on this, sequences of fuzzy numbers have been introduced and investigated by many authors , they studied important pro - perties. The works on double sequence of real numbers in Bromwich , was found [2]. Moreover the double sequence was exacted by Moricz [3], Basarir and Sonalncan [4], Tripathly and Sarma [5] and many others.

The notion of difference sequence space of complex numbers was introduced by Kizmaz [6].

On this idea different type of difference sequence spaces was introduced and studied their properties by Tripathy and Baruah [7],Tripathy and Borgohain [8,9],Tripathy and Mahanta [10].

Battor , Neamah [13] defined the double sequence by a double orlicz function $M(X,Y)$=$(M_1(X),M_2(Y))$ and introduced vector valued double sequence spaces over semi-normed space

2. Definitions and Preliminaries

Tripathy and Borgohain [8] defined the space $C(R^n)$ and the Hausdorff distance on the usual Euclidean R^n, we defined they on R^{2n}(the real Euclidean $2n$-space of dimension $2n$) as follows:

Let $C(R^{2n})=\{A\subset R^{2n}: A$ is compact and convex $\}$, then this space has linear structure induced by the following operations:

$A+B=\{(x,y)+(z,w):(x,y)\in A,(z,w)\in B\}$ and $\lambda A=\{\lambda(x,y):(x,y)\in A\}$ for $A,B \in C(R^{2n})$ and $\lambda\in R$.Define the Hausdorff distance between A and B of $C(R^{2n})$ as follows:

$$d(A,B)=max\{sup_{(x,y)\in A}inf_{(z,w)\in B}\|(x,y)-(z,w)\|,sup_{(z,w)\in B}inf_{(x,y)\in A}\|(x,y)-(z,w)\|\},$$

where $\|.\|$ denotes to the usual Euclidean norm in R^{2n}. It is well known that $(C(R^{2n})$,d) a complete metric space.

Now, fuzzy real number X on R^{2n} is a function $X: R^{2n}\to I=[0,1]$ associating for all real number t $\in R^{2n}$, with its grade to membership $X(t)$.The class of all fuzzy real numbers is denoted by $R^{2n}(I)$. For $0<\alpha\le 1$, α-level set $X^\alpha=\{t\in R^{2n}: X(t)\ge\alpha\}$ is a nonempty compact convex, subsets of R^{2n}, as the support $X^0=\lim_{\alpha\to 0^+}X^\alpha$. A linear structure of $C(R^{2n})$ induces the addition $X+Y$ and scalar multiplication λX, $\lambda\in R$ in terms of α-level sets, by $[X+Y]^\alpha=[X]^\alpha+[Y]^\alpha$ and $[\lambda X]^\alpha=\lambda[X]^\alpha$, for $0<\alpha\le 1$, such that

$$\lambda X(t)=\begin{cases}\bar{0}, & \text{for } \lambda=0\\ X(\lambda^{-1}t), & \text{otherwise.}\end{cases}$$

Define a map $\bar{d}: R^{2n}(I)\times R^{2n}(I)\to R$ for $X,Y,F,H\in R^{2n}$(I) such that $\bar{d}\big((X,Y),(F,H)\big)=sup_{0<\alpha\le 1}(d(X^\alpha,Y^\alpha),\quad d(F^\alpha,H^\alpha))$

, where d is the Hausdorff metric. $\big(R^{2n}(I),\bar{d}\big)$ is a complete, locally compact and separable metric space. The additive and multiplicative identities of $R^{2n}(I)$ are denoted by $\bar{\theta}$ and $\bar{e}$, respectively, such that $\theta=\big((0,0),(0,0),\dots,(0,0)\big)$ and $e=\big((1,1),(1,1),\dots,(1,1)\big)$.The zero double sequence of fuzzy real numbers is denoted by $\bar{\vartheta}=\{(\bar{\theta},\bar{\theta}),(\bar{\theta},\bar{\theta}),\dots,(\bar{\theta},\bar{\theta}),\dots\}$.

Throughout the article 2E represent a semi-normed space by a semi norm q [15].

Definition 2.1. A double sequence $(X,Y)=(X_{rs},Y_{rs})$ on fuzzy real numbers is said to be *converge* to the fuzzy numbers X_{00},Y_{00}, if for every $\varepsilon>0$, there exists r_o, $s_o\in N$ such that $\bar{d}(X_{rs},X_{00})<\varepsilon$, $\bar{d}(Y_{rs},Y_{00})<\varepsilon$ for all $r\geq r_o$, $s\geq s_o$, and consequently $\bar{d}((X_{rs},X_{00}),(Y_{rs},Y_{00}))<\varepsilon$, for all $r\geq r_o$, $s\geq s_o$.

Definition 2.2. A double sequence $(X,Y)=(X_{rs},Y_{rs})$ on fuzzy real numbers is said to be a *double Cauchy sequence*, if for every $\varepsilon>0$ there exists $n_0\in N$ such that $\bar{d}(X_{ij},X_{rs})<\varepsilon,\bar{d}(Y_{ij},Y_{rs})<\varepsilon$, to all $i\geq r\geq n_0, j\geq s\geq n_0$, and consequently $\bar{d}\left((X_{ij},X_{rs}),(Y_{ij},Y_{rs})\right)<\varepsilon$, to all $i\geq r\geq n_0, j\geq s\geq n_0$.

Definition 2.3. A double sequence space $2E$ is said to be *solid* if $(F_{rs},H_{rs})\in 2E$, whenever $(X_{rs},Y_{rs})\in 2E$ and $|(F_{rs},H_{rs})|\leq|(X_{rs},Y_{rs})|$, to all $r,s\in N$.

Definition 2.4. A double sequence space $2E$ is said to be *symmetric* if $S(X_{rs},Y_{rs})\subset 2E$ whenever $(X_{rs},Y_{rs})\in 2E$, where $S(X_{rs},Y_{rs})$ means the set of all permutations of the elements of (X_{rs},Y_{rs}) it is ,$S(X_{rs},Y_{rs})=\{(X_{\pi(r),\pi(s)},Y_{\pi(r),\pi(s)}):\pi \text{ is permutation on N}\}$.

Definition 2.5. The double sequence space $2E$ is said to be *convergence-free* if $(\mathrm{F}_{rs},\mathrm{H}_{rs})\in 2E$ whenever $(X_{rs},Y_{rs})\in 2E$ and $(X_{rs},Y_{rs})=(\bar{\theta},\bar{\theta})$ implies that

$(\mathrm{F}_{rs},\mathrm{H}_{rs})=(\bar{\theta},\bar{\theta})$.

Definition 2.6. A double sequence space $2E$ is said to be *monotone*. if $2E$ contains the canonical preimages of all its step spaces.

Lemma 2.7 The class of double sequences $2E$ is *solid* implies that $2E$ is *monotone*.

Lindenstrauss and Tzafrir [14] used the idea of Orlicz function and the sequence space L_M of single sequences, later Battor and Neamah [12] aplied that idea to construct this double sequence space:

$$2L_M=\left\{(X_{rs},Y_{rs})\in 2\omega:\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\left(M_1\left(\frac{|X_{rs}|}{\rho}\right)\right)\vee\left(M_2\left(\frac{|Y_{rs}|}{\rho}\right)\right)\right]<\infty,\text{ for some }\rho>0\right\},$$

which is Banach space under the norm:

$$\|(X_{rs},Y_{rs})\|_M=\inf\left\{\rho>0:\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\left(M_1\left(\frac{|X_{rs}|}{\rho}\right)\right)\vee\left(M_2\left(\frac{|Y_{rs}|}{\rho}\right)\right)\right]\leq 1\right\},$$

which is said a double Orlicz of a double sequence space where 2ω is a family of all $\mathbb{R}^2$ or $\mathbb{C}^2$ double sequence, so (X_{rs}) and (Y_{rs}) are complex or real double sequence and epilogue that the double Orlicz of a double sequence space $2L_M$ be neatly related to the space $2L_p$, which is a double Orlicz double sequence space with $M(X,Y)=(M_1(X),M_2(Y))=(X^p,Y^p)$, for $(1,1)\leq(p,p)<(\infty,\infty)$ such that $M_1(X)=X^p$, for $1\leq p<\infty$, and $M_2(Y)=Y^p$, for $1\leq p<\infty$.

Tripathy et al. [12] generalized difference sequence spaces, later Leena, Ali and Maysoon [16] used that concept to generalized motif the difference double sequence space $Z(\Delta_m^n)$,to $m\geq 1,n\geq 1$,

In this paper we introduce the following a semi-normed difference double sequence spaces of fuzzy real numbers, for a double Orlicz function:

$$(2\ell_\infty)^F(M,\Delta_m^n)=$$

$$\left\{(X_{rs},Y_{rs}) \in (2\omega)^F: sup_{r,s}\left[\left(M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},\bar{\theta})}{\rho}\right)\right)\right)\bigvee\left(M_2\left(\left(\left(\frac{\bar{d}(\Delta_m^n Y_{r,s},\bar{\theta})}{\rho}\right)\right)\right)\right] < \infty, \text{for some } \rho > 0\right\},$$

$$(2c)^F(M,\Delta_m^n) = \left\{(X_{rs},Y_{rs}) \in (2\omega)^F : \lim_{r,s\to\infty}\left[\left(M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},\mathrm{L}_1)}{\rho}\right)\right)\right)\vee\left(M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},\mathrm{L}_2)}{\rho}\right)\right)\right)\right] = 0, \text{ for some } \rho > 0,\ \mathrm{L}_1,\mathrm{L}_2 \in R^{2n}(I)\right\},$$

and

$$(2c_0)^F(M,\Delta_m^n) = \left\{(X_{rs},Y_{rs}) \in (2\omega)^F: \lim_{r,s\to\infty}\left[\left(M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},\bar{\theta})}{\rho}\right)\right)\right)\bigvee\left(M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},\bar{\theta})}{\rho}\right)\right)\right)\right] = 0, \text{for some } \rho > 0\right\}$$

3. Main Results

Theorem 3.1 Suppose a semi-normed space (X^2,q) which is complete, the classes of double sequences $(2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_m^n)$, $(2c_0)^F(M,\Delta_m^n)$ are complete semi-normed spaces by

$$g_M\big((X,Y),(F,H)\big) = \sum_{k,l=1}^{mn}\bar{d}\big((X_{kl},Y_{kl}),(F_{kl},H_{kl})\big) +$$

$$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},\Delta_m^n Y_{rs})}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n F_{rs},\Delta_m^n H_{rs})}{\rho}\right)\right)\right] \le 1\right\},$$

for $M\big((X,Y),(F,H)\big) = \big(M_1(X,Y),M_2(F,H)\big)$, and $(X,Y),(F,H) \in (2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_m^n)$, $(2c_0)^F(M,\Delta_m^n)$.

Proof. We prove the result for $(2\ell_\infty)^F(M,\Delta_m^n)$. The proof for the other cases is similarly. Clearly $(2\ell_\infty)^F(M,\Delta_m^n)$ is a semi-normed space of g . Next we prove it is a complete semi-normed space .

Let (X^i,Y^i) be a double Cauchy sequence in $(2\ell_\infty)^F(M,\Delta_m^n)$, such that $(X^i),(Y^i)$ be a Cauchy sequence in $(2\ell_\infty)^F(M_1,\Delta_m^n)$, $(2\ell_\infty)^F(M_2,\Delta_m^n)$, respectively, so $(X^i) = \left(X_{nq}^i\right)_{n,q=1}^\infty$, $(Y^i) = \left(Y_{nq}^i\right)_{n,q=1}^\infty$. Let $\varepsilon > 0$ be given, for a fixed $\eta > 0$, select $t > 0$ for example $M_1\left(\frac{t\eta}{2}\right) \ge 1$, $M_2\left(\frac{t\eta}{2}\right) \ge 1$, that is, $M\left(\frac{t\eta}{2},\frac{t\eta}{2}\right) = \left(M_1\left(\frac{t\eta}{2}\right),M_2\left(\frac{t\eta}{2}\right)\right) \ge (1,1)$. Thus it exists a positive integer $n_0(\varepsilon)$ such $g_{M_1}(X^i,X^j) < \frac{\varepsilon}{t\eta}$ and $g_{M_2}(Y^i,Y^j) < \frac{\varepsilon}{t\eta}$, $\forall\ i,j \ge n_0$, and consequently, $g_M\left((X^i,X^j),(Y^i,Y^j)\right) < \frac{\varepsilon}{t\eta}$, $\forall\ i,j \ge n_0$. Then By definition of g, we obtain that, $\sum_{k=1}^{mn}\sum_{l=1}^{mn}\bar{d}\left((X_{kl}^i,X_{kl}^j),(Y_{kl}^i,Y_{kl}^j)\right) +$

$$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}\left(\Delta_m^n X_{rs}^i,\Delta_m^n X_{rs}^j\right)}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}\left(\Delta_m^n Y_{rs}^i,\Delta_m^n Y_{rs}^j\right)}{\rho}\right)\right)\right] \le 1\right\} \le \varepsilon, \quad \dots(3-1)$$

$\forall\ i,j \ge n_0$, which implies $\sum_{k=1}^{mn}\sum_{l=1}^{mn}\bar{d}\left((X_{kl}^i,X_{kl}^j),(Y_{kl}^i,Y_{kl}^j)\right) < \varepsilon$ $\forall \mathrm{i,j} \ge n_0$.

Hence, $\bar{d}\left((X_{kl}^i,X_{kl}^j),(Y_{kl}^i,Y_{kl}^j)\right) < \varepsilon$, $\forall \mathrm{i,j} \ge n_0$, $k,l = 1,2,3,\dots,mn$. Thus (X_{kl}^i,Y_{kl}^i), for

$k,l = 1,2,3,\dots,mn$ is double Cauchy sequence in $R^{2n}(I)$, hence it is convergent in $R^{2n}(I)$ by the completeness monarchy of $R^{2n}(I)$.

Let $\lim_{i\to\infty}(X_{kl}^i, Y_{kl}^i) = (X_{kl}, Y_{kl})$, to $k, l = 1,2,3, \ldots, mn$. $\ldots(3-2)$

Also, $\sup_{r,s}\left[M_1(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j)}{\rho}\right)) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j)}{\rho}\right)\right)\right] \leq 1$, $\forall$i,j $\geq n_0$, $\ldots(3-3)$

$$\Rightarrow \left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j)}{g_{M_1}(X^i, X^j)}\right)\right), M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j)}{g_{M_2}(Y^i, Y^j)}\right)\right)\right] \leq (1,1) \leq \left(M_1\left(\frac{t\eta}{2}\right), M_2\left(\frac{t\eta}{2}\right)\right), \forall \text{i,j} \geq n_0.$$

By continuity of M so M_1, M_2 , we have

$\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j) \leq \frac{t\eta}{2} . g_{M_1}(X^i, X^j)$ and $\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j) \leq \frac{t\eta}{2} . g_{M_2}(Y^i, Y^j)$, $\forall$i,j $\geq n_0$,

$\Rightarrow \bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j) \leq \frac{t\eta}{2} . \frac{\varepsilon}{t\eta} = \frac{\varepsilon}{2}$, $\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j) < \frac{t\eta}{2} . \frac{\varepsilon}{t\eta} = \frac{\varepsilon}{2}$, $\forall$i,j $\geq n_0$, so we get $\bar{d}\left((\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j), (\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j)\right) < \frac{t\eta}{2} . \frac{\varepsilon}{t\eta} = \frac{\varepsilon}{2}$, $\forall$i,j $\geq n_0$,which implies $(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i)$ is double Cauchy sequence of $R^{2n}(I)$, so is convergent in $R^{2n}(I)$ by a completeness property of $R^{2n}(I)$.

we have to prove that, $\lim_i (X^i, Y^i) = (X, Y)$, $(X, Y) \in (2\ell_\infty)^F(M, \Delta_m^n)$.

Let $\lim_i(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i) = (F_{rs}, H_{rs})$ (say), in $R^{2n}(I)$, for all $r, s \in N$.

To r, s =1, we have, from $(1-1)$, $(3-2)$,

$\lim_{i\to\infty}(X_{mn+1,mn+1}^i, Y_{mn+1,mn+1}^i) = (X_{mn+1,mn+1}, Y_{mn+1,mn+1})$, for $m \geq 1$,$n \geq 1$.

Continuance in this way , $\lim_{i\to\infty}(X_{rs}^i, Y_{rs}^i) = (X_{rs}, Y_{rs})$,for all $r, s \in N$.

Also, $\lim_i(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i) = (\Delta_m^n X_{rs}, \Delta_m^n Y_{rs})$, for each $r, s \in N$.

Thus, for using the continuity of M, then M_1, M_2 and takeover $j \to \infty$, let i fixed, that have following from $(3-3)$:

$$\sup_{r,s}\left[M_1(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right)) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right)\right] \leq 1, \quad \text{for some } \rho > 0.$$

Now, if we take a infimum of such $\rho's$ and using $(3-1)$, we get

$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right)\right) \vee M_2(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right))\right] \leq 1\right\} < \varepsilon$, $\forall$i $\geq n_0$. Thus on going to limit as $j \to \infty$ and using $(3-1)$ again ,we get

$$\sum_{k=1}^{mn}\sum_{l=1}^{mn} \bar{d}\left((X_{kl}^i, X_{kl}), (Y_{kl}^i, Y_{kl})\right) +$$
$$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right)\right] \leq 1\right\} < \varepsilon + \varepsilon = 2\varepsilon, \ \forall i \geq n_0 ,$$

which implies that $g\left((X^i, X), (Y^i, Y)\right) < 2\varepsilon$, $\forall$i $\geq n_0$. That is, $\lim_i(X^i, Y^i) = (X, Y)$.

Now, we show that $(X, Y) \in (2\ell_\infty)^F(M, \Delta_m^n)$, we have,

$$\bar{d}\left((\Delta_m^n X_{rs}, \bar{\theta}), (\Delta_m^n Y_{rs}, \bar{\theta})\right) \leq \bar{d}\left((\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}), (\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})\right) + \bar{d}\left((\Delta_m^n X_{rs}^i, \bar{\theta}), (\Delta_m^n Y_{rs}^i, \bar{\theta})\right),$$

By continuity of M so M_1, M_2, also it's non-decreasing , then we get

$$\sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}, \bar{\theta})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}, \bar{\theta})}{\rho}\right)\right)\right]$$
$$\leq \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right)\right] +$$
$$\sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \bar{\theta})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \bar{\theta})}{\rho}\right)\right)\right] < \infty,$$

which implies that $(X, Y) \in (2\ell_\infty)^F(M, \Delta_m^n)$. Hence $(2\ell_\infty)^F(M, \Delta_m^n)$ is complete semi-normed space .

***Result* 1.** The classes of a double sequences $(2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_m^n)$, $(2c_0)^F(M,\Delta_m^n)$ are not *symmetric*.

***Proof*.** The effect follows from following example.

***Example* 3.2** Let $m=2, n=1$ and $M(X,Y)=\left(M_1(X),M_2(Y)\right)=(X^2,Y^2)$, $\forall\ (X,Y)\in[0,\infty)\times[0,\infty)$.

Considerthe double sequence (X_{rs},Y_{rs}) define by

$$(X_{rs},Y_{rs})(t)=\begin{cases}(1,1), & \text{for } r,s\in N, t=(rs,rs,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$$

Then, $(\Delta_2 X_{rs},\Delta_2 Y_{rs})(t)=\begin{cases}(1,1), & \text{for } r,s\in N, t=(-2,-2,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$

Then, $\bar{d}\left((\Delta_2 X_{rs},\bar{0}),(\Delta_2 Y_{rs},\bar{0})\right)=(1,1)$, for all $r,s\in N$, which shows $(X_{rs},Y_{rs})\in (2c)^F(M,\Delta_2)\subset(2\ell_\infty)^F(M,\Delta_2)$.Now, let (F_{rs},H_{rs}) be a rearrangement of (X_{rs},Y_{rs}) defined as $(F_{rs},H_{rs})=\left((X_{11},Y_{11}),(X_{22},Y_{22}),(X_{44},Y_{44}),(X_{33},Y_{33}),(X_{99},Y_{99}),(X_{55},Y_{55}),(X_{16},Y_{16}),\ldots\right)$.

Then we get, $\bar{d}\left((\Delta_2 F_{rs},\bar{0}),(\Delta_2 H_{rs},\bar{0})\right)\approx(rs-(rs-1)^2,rs-(rs-1)^2)\approx(r^2s^2,r^2s^2)$,for all r, s $\in$N, which implies, $\sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_2 F_{rs},\bar{0})}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_2 H_{rs},\bar{0})}{\rho}\right)\right)\right]=\infty$, for each fixed $\rho>0$. Hence,$(F_{rs},H_{rs})\notin (2\ell_\infty)^F(M,\Delta_2)$. So the classes of sequences$(2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_2)$ and $(2c_0)^F(M,\Delta_m^n)$ are not symmetric.

Notes. i) If $m=n=0$, then $(2\ell_\infty)^F(M)$ and $(2c)^F(M)$ are *symmetric*.

ii) If $m\le 1, n\le 1$, then $(2c_0)^F(M,\Delta_m^n)$ is *symmetric*.

***Result* 2.** The classes of double sequences $(2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_m^n)$, $(2c_0)^F(M,\Delta_m^n)$ are not *solid*, so not *monotone*.

***Proof*.** We will take this following example . $M(X,Y)=\left(M_1(X),M_2(Y)\right)=$

Example 3.3 Let $m=2$, $n=3$ and $M(X,Y)=(|X|,|Y|)$, for all $(X,Y)\in[0,\infty)\times[0,\infty)$.

Consider the double sequence (X_{rs},Y_{rs}) defined as

$$(X_{rs},Y_{rs})(t)=\begin{cases}(1,1), & \text{for } r,s\in N,\ t=(rs,rs,rs,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$$

Then,$(\Delta_2^3 X_{rs},\ \Delta_2^3 Y_{rs})(t)=\begin{cases}(1,1), & \text{for } r,s\in N,\ t=(0,0,0,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$

$\Rightarrow\ \bar{d}\left((\Delta_2^3 X_{rs},\bar{\theta}),(\Delta_2^3 Y_{rs},\bar{\theta})\right)=(0,0)$, for $r,s\in N$. Hence, we have

$\sup_{rs}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_2^3 X_{rs},\bar{\theta})}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_2^3 Y_{rs},\bar{\theta})}{\rho}\right)\right)\right]<\infty$, for some $\rho>0$.This implies$(X_{rs},Y_{rs})\in (2\ell_\infty)^F(M,\Delta_2^3)$. Now, let (α_{rs},β_{rs}) be a double sequence of scalars defined as

$$(\alpha_{rs},\beta_{rs})=\begin{cases}(1,1), & \text{for } r,s=i^2, i\in N,\\ (0,0), & \text{otherwise.}\end{cases}$$

Then, for $r,s=i^2$, we have

$$(\alpha_{rs}X_{rs},\beta_{rs}Y_{rs})(t)=\begin{cases}(1,1), & t=(rs,rs,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$$

For $r,s\neq i^2$, we have

$$(\alpha_{rs}X_{rs},\beta_{rs}Y_{rs})(t)=\begin{cases}(1,1), & t=(0,0,\ldots),\\ (0,0), & \text{otherwise.}\end{cases}$$

$$\Rightarrow \quad \sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_2^3 X_{rs},\bar{\theta})}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_2^3 Y_{rs},\bar{\theta})}{\rho}\right)\right)\right] = \infty, \text{ for some } \rho > 0 .$$

Thus $(\alpha_{rs}X_{rs},\beta_{rs}Y_{rs}) \notin (2\ell_\infty)^F(M,\Delta_2^3)$. Hence,$(2\ell_\infty)^F(M,\Delta_m^n)$ is not *solid*, so not *monotone* by lemma 1.Similarly,the other classes can be proven.

Proposition 3.4 These double sequences $(2\ell_\infty)^F(M,\Delta_m^n)$, $(2c)^F(M,\Delta_m^n)$,$(2c_0)^F(M,\Delta_m^n)$ aren't *convergence-free.*

Proof. To show, that we have the following example.

Example 3.5 Let $m = 4$ and $n = 1, M(X,Y) = (X^3,Y^3)$, to all $(X,Y) \in [0,\infty)\times[0,\infty)$ and consider the double sequence (X_{rs},Y_{rs}) defined as

$$(X_{rs},Y_{rs})(t) = \begin{cases} (1,1), & \text{for } r,s \in N, t = \left(\frac{1}{rs},\frac{1}{rs},\dots\right) \\ (0,0), & \text{otherwise.}\end{cases}$$

Then $(\Delta_4 X_{rs},\Delta_4 Y_{rs})(t) = \begin{cases}(1,1), & \text{for } r,s \in N, t = \left(\frac{4}{rs(rs+4)},\frac{4}{rs(rs+4)},\dots\right) \\ (0,0), & \text{otherwise}\end{cases}$.

$\Rightarrow \bar{d}\left((\Delta_4 X_{rs},\bar{\theta}),(\Delta_4 Y_{rs},\bar{\theta})\right) = \left(\frac{4}{rs(rs+4)},\frac{4}{rs(rs+4)}\right)$, which implies $(X_{rs},Y_{rs}) \in (2c_0)^F(M,\Delta_4)$ $\subset (2c)^F(M,\Delta_4) \subset (2\ell_\infty)^F(M,\Delta_4)$. Now, let (F_{rs},H_{rs}) be a double sequence defined as

$$(F_{rs},H_{rs})(t) = \begin{cases}(1,1), & \text{for } r,s \in N, t = ((rs)^2,(rs)^2,\dots) \\ (0,0), & \text{otherwise.}\end{cases}$$

Furthermore, $(\Delta_4 F_{rs},\Delta_4 H_{rs})(t) = \begin{cases}(1,1), & \text{for } r,s \in N, t = (-(8rs+16),-(8rs+16),\dots) \\ (0,0), & \text{otherwise,}\end{cases}$

$\Rightarrow \bar{d}\left((\Delta_4 F_{rs},\bar{\theta}),(\Delta_4 H_{rs},\bar{\theta})\right) = (8rs+16, 8rs+16)$, for $r,s \in N$, which implies, $\sup_{r,s}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_4 F_{rs},\bar{\theta})}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_4 H_{rs},\bar{\theta})}{\rho}\right)\right)\right] = \infty$, for some $\rho > 0$.This means, $(F_{rs},H_{rs}) \notin (2\ell_\infty)^F(M,\Delta_4)$.

Hence the classes $(2\ell_\infty)^F(M,\Delta_m^n)$,$(2c)^F(M,\Delta_m^n)$, $(2c_0)^F(M,\Delta_m^n)$ are not *convergence-free.*

Theorem 3.6 Let $M = (M_1,M_2)$ and $\mathcal{M} = (M_3,M_4)$ be a double Orlicz functions satisfying Δ_2-condition. Then, for $Z = (2\ell_\infty)^F,(2c)^F,(2c_0)^F$,

(***i***) $Z(M,\Delta_m^n) \subseteq Z(\mathcal{M}\circ M,\Delta_m^n)$,

(***ii***) $Z(M,\Delta_m^n) \cap Z(\mathcal{M},\Delta_m^n) \subseteq Z(M+\mathcal{M},\Delta_m^n)$.

Proof. We show this result for the space $(2\ell_\infty)^F(M,\Delta_m^n)$. The other spaces can be shown similarly.

(***i***) Let $(X_{rs},Y_{rs}) \in (2\ell_\infty)^F(M,\Delta_m^n)$ such that $X_{rs} \in (2\ell_\infty)^F(M_1,\Delta_m^n)$, $Y_{rs} \in (2\ell_\infty)^F(M_2,\Delta_m^n)$.

Consider $\varepsilon > 0$ and $\tau > 0$ such that $\varepsilon = \mathcal{M}(\tau)$.

Then $\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L)}{\rho}\right)\right), M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L)}{\rho}\right)\right)\right] < \tau$,

for some $\rho > 0$.Let $(F_{rs},H_{rs}) = \left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L)}{\rho}\right)\right), M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L)}{\rho}\right)\right)\right]$, for some $\rho > 0$.

We know $\mathcal{M}$ is continuous and non-decreasing , then we have,

$$\mathcal{M}(F_{rs},H_{rs}) = \mathcal{M}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L)}{\rho}\right)\right)\vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L)}{\rho}\right)\right)\right] < \mathcal{M}(\tau) = \varepsilon \text{ , for some } \rho > 0 \text{ ,}$$

which means $(X_{rs},Y_{rs}) \in Z(\mathcal{M}\circ M,\Delta_m^n)$. Hence the proof is complete.

(***ii***) Let $(X_{rs},Y_{rs}) \in Z(M,\Delta_m^n) \cap Z(\mathcal{M},\Delta_m^n)$. Then for some $\rho > 0$, we have,

$$\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho}\right)\right),M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right)\right] < \varepsilon\ ,\ \sup_{r,s}\left[M_3\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho}\right)\right),M_4\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right)\right] < \varepsilon.$$

From the equality the proof follows:

$$(M+\mathcal{M})\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho},\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right) = \left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho}\right)\right),M_2\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right)\right] + \left[M_3\left(q\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho}\right)\right),M_4\left(q\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right)\right] < \varepsilon+\varepsilon = 2\varepsilon,$$

to some $\rho > 0$ that means $(X_{rs},Y_{rs}) \in Z(M+\mathcal{M},\Delta_m^n)$. Hence the proof is complete.

Proposition 3.7 One has $Z(M,\Delta_m^i) \subset Z(\ M,\Delta_m^n)$, for $0 \le i < n$ where $M = (M_1,M_2)$

And $Z = (2\ell_\infty)^F,(2c)^F,(2c_0)^F$.

Proof.

Let $(X_{rs},Y_{rs}) \in (2\ell_\infty)^F(M,\Delta_m^{n-1})$ such that $X_{rs} \in (2\ell_\infty)^F(M_1,\Delta_m^{n-1})$, $Y_{rs} \in (2\ell_\infty)^F(M_2,\Delta_m^{n-1})$. Then, we have $\left(sup_{r,s\ge1}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}X_{rs},\bar{0})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{rs},\bar{0})}{\rho}\right)\right)\right]\right) < \infty$.

We have, $\left(sup_{r,s\ge1}\left[M_1 q\left(\frac{\bar{d}(\Delta_m^{n-1}X_{rs}-\Delta_m^{n-1}X_{r,s+m},\bar{0})}{\rho}\right) \vee M_2 q\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{rs}-\Delta_m^{n-1}Y_{r,s+m},\bar{0})}{\rho}\right)\right]\right)$

$$\le \tfrac{1}{2}sup_{r,s\ge1}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}X_{rs},\bar{0})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{rs},\bar{0})}{\rho}\right)\right)\right] + \tfrac{1}{2}sup_{r,s\ge1}\left[M_1\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}X_{r,s+m},\bar{0})}{\rho}\right)\right) \vee M_2\left(q\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{r,s+m},\bar{0})}{\rho}\right)\right)\right] < \infty.$$

Follow up in this way, we have $Z(M,\Delta_m^i) \subset Z(\ M,\Delta_m^n)$, for $0 \le i < n$. This complete the proof.

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