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SOLVING (2+1) RIEMANN WAVE EQUATION VIA FIRST INTEGRAL METHOD

Susan H. MOHAMMAD[1]
University of Mosul, Iraq

Merna A. SAMARCHI[2]
University of Mosul, Iraq

Ekhlass S. AL-RAWI[3]
University of Mosul, Iraq

Abstract

This research sought to find an approximate solution to the partial differential equations (PDEs) by relying on a method based on the commutative algebra theory, and this method is (the first integral method FIM). At first, the order of the partial differential equation was reduced to an ordinary differential equation through traveling wave transformations, then it was solved using the first soft steps of integration, as it produced the approximate solutions to the problem. This method is characterized by being brief, direct and effective when finding the approximate solution to PDEs.

Keywords: *First integral method, Riemann wave equation, Traveling wave transformations.*

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[1] susan.al-hakam@uomosul.edu.iq, https://orcid.org/0000-0002-8129-4367
[2] merna_samarchi@uomosul.edu.iq, https://orcid.org/0000-0003-3909-2339
[3] drekhlass-alrawi@uomosul.edu.iq, https://orcid.org/0000-0001-6940-7050

Introduction

The study's goal was to investigate the nonlinear Riemann wave equation(RWE), which represents tidal waves, tsunamis, and other waves in stationary, homogeneous mediums. Because of the recent climate changes and increasing fears of tidal waves and tsunamis, researchers were interested in studying and finding accurate solutions to the RWE. For example, Hemonta K. B. presented in[1] the solution using the generalized Kudryashov method. The wave hypothesis has also been utilized, together with the expanded Tanh approach in[2]. S. Z. Majid et al. investigated and analyzed the soliton theory-based analytical solutions to the RWE`. The novel extended direct algebraic method was used to discuss the explicit soliton structures[3] .

Regarding the solution approach used in our study, it is the FIM, Feng [4]presented the method by utilizing loop theory in commutative algebra. The technique has attracted the attention of many academics since it has been successfully used to solve several nonlinear PDEs and offers explicit accurate solutions without requiring laborious calculations. The method was used to explore some new exact solutions to Gardener's equation and the Sharma-Tasso-Over equation in [5]**.** Three nonlinear fractional partial differential equations The modified Zakharov-Kuznetsov equations, the modified Benjamin-Bona-Mahony equations, and the Whitham-Broer-Kaup-Like equations were solved satisfactorily using the approach [6]. The generalized Zakharov system and the negative order KdV equation were solved in[7]. Alquran et al. focused on nonlinear system of PDEs with both real and complex-valued unknown functions. The linked Higgs field equations, the Davey-Sterwatson equations, and the coupled Klein-Gordon-Zakharov equations[8]. The first integral approach has been used in [9] to examine three nonlinear PDEs Burgers-Fisher, Burgers-Huxley, and modified Korteweg-de Vries). N. Taghizadeh and M. Mirzazadeh solved the coupled Higgs field problem and Hamiltonian amplitude equation[10]. In [11] they investigated the method for Maccari's system solutions. The exact solutions for the nonlinear differential equations were found for the Broer-Kaup equations, and approximations for long water wave equations were also found. [12]. In [13]the method found the exact solutions of the Eckhaus equation. The same method was used to create traveling wave solutions for the Broer-Kaup (BK) and Whitham Broer-Kaup (WBK) systems[14]. The perturbed Klein-Gordon equation , the Klein-Gordon-Zakharov equations, the Drinfeld-Sokolov system, and the perturbed Klein-Gordon equation are all solved using the first integral method in[15]. A. Bekir and Ö. Ünsal demonstrated in their study how the combined KdV-mKdV equation, the Pochhammer-Chree equation, and coupled nonlinear evolution equations [16]. Two higher-order nonlinear Schrödinger equations have soliton wave solutions, kink wave solutions, and periodic wave solutions. The results show how helpful this strategy is for obtaining accurate solutions[17].

In this work, the Riemann wave problem is solved using the first integral method. The work was organized into the following steps: At first, the basic steps required to solve the problem were presented in the proposed way, step by step, and then the aforementioned issue was solved and the possibilities resulting from the solution were explored.

1. Steps explanation:

Before explaining the steps, we shall state the Division Theorem:

Division Theorem: Suppose that $P(x, y)$ and $Q\ (x, y)$ are polynomials of two variables x and y and P(x, y), is irreducible in C[x, y]. If $Q\ (x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial T (x, y) in C[x, y] such that

$$Q\ (x, y)\ =\ P(x, y)T\ (x, y)$$

The Division Theorem follows immediately from the Hilbert–Nullstellensatz Theorem [18]

Consider the nonlinear system of PDE:

$$P_1(U, V, U_t, V_t, U_x, V_x, U_{tt}, V_{tt}, U_{xx}, V_{xx} \dots) = 0$$

$$P_2(U, V, U_t, V_t, U_x, V_x, U_{tt}, V_{tt}, U_{xx}, V_{xx} \dots) = 0$$

(1)

Step 1. Applying the transformations $u(x, t)\ =\ f\ (\omega)$ and $v(x, t)\ =\ g\ (\omega)$ where $\omega = ax - ct$ converts Equations system(1) into a system of ordinary differential equations:

$$Q_1(f, g. f', g', \dots) = 0$$

(2)

$$Q_2(f, g. f', g', \dots) = 0$$

(3)

Where prime denotes the derivative with respect to the same variable ω .

Step 2. Using the mathematical operations, the system is converted to a second – order as:

$$R_2(f, g. f', g', f'', g'' \dots) = 0$$

(4)

Step 3. By introducing new variables $X(\omega) = f(\omega)$ and $Y(\omega) = \frac{\partial f(\omega)}{\partial \omega}$ the system take the form:

$$X'(\omega) = Y(\omega)$$

(5)

$$Y'(\omega) = K(X, Y)$$

(6)

Step 4. The Division Theorem, which is based on the ring theory of commutative algebra, was applied to Equations 5 and 6, which are reduced to a first-order integrable ordinary differential equation. Finally, an approximate solution to Equation (1) was found by solving the resulting first-order integrable differential equation.

3. Application

Bernhard Riemann's study "On the propagation of plane waves of finite amplitude" (Riemann, 1860), which served as the first mathematical analysis of the gas dynamics equations, was published in 1860. A study of the Riemann problem, a hyperbolic partial differential equation (PDE) [19]. In this study, we aim to estimate its solution using the first integral method. Consider the Riemann wave equation:

$$U_t + pU_{xxy} + \mu U W_x + \gamma W U_x = 0$$

(7)

$$W_x = U_y$$

(8)

Where p , μ and γ are parameters .

By the traveling wave transformation, we suppose that:

$$U(x, y, t) = U(\omega) \quad , \quad \omega = ax + by - ct$$

(9)

Transforming Equation (1) into ordinary differential equation, holed:

$$-cU' + pa^2bU''' + \mu aUW' + \gamma aU'W = 0$$

(10)

$$W' = \frac{b}{a}U'$$

(11)

Integrating Equation(11) once with respect to (ω) and let the integration constant be (0) , we get:

$$W = \frac{b}{a}U$$

(12)

Substituting (W) in Equation (10), it becomes:

$$-cU' + pa^2bU''' + \mu aU(\frac{b}{a}U)' + \gamma aU'\frac{b}{a}U = 0$$

(13)

Then Equation (13) becomes:

$$-cU' + pa^2bU''' + \mu bUU' + \gamma bU'U = 0$$

(14)

So

$$pa^2bU''' + b(\mu+\gamma)UU' - cU' = 0$$

(15)

By integrating Equation (15) with respect to (ω) and supposing that the integration constant is (0), the following results are obtained:

$$pa^2bU'' + b(\mu+\gamma)U^2 - cU = 0$$

(16)

Making the transformation $U(x,y,t) = f(\omega) = f(ax+by-ct)$, the Equation(16) becomes:

$$pa^2bf'' + b(\mu+\gamma)f^2 - cf = 0$$

(17)

$$f'' = \frac{1}{pa^2b}(cf - b(\mu+\gamma)f^2$$

(18)

Suppose that $X(\omega) = f(\omega)$ and $Y(\omega) = \frac{\partial f(\omega)}{\partial \omega}$ the Equation(11) is yield to :

$$X'(\omega) = Y(\omega)$$

(19)

$$Y'(\omega) = \frac{1}{pa^2b}(cf - b(\mu+\gamma)f^2)$$

(20)

The hypotheses $X(\omega)$ and $Y(\omega)$ are nontrivial solutions to Equations (19 & 20).

According to the FIM , let $q(X,Y)$ be an irreducible polynomial in the complex domain C[X,Y], such that :

$$q\big(X(\omega), Y(\omega)\big) = \sum_{i=0}^{m} k_i(X)Y^i = 0$$

(21)

Noting that $k_m(X) \neq 0$ where $k_i(X)$ for $(i = 0,1,\dots,m)$ are polynomials of X .

Equation(21) is called the FIM to Equations (19 & 20).Du to Division Theorem ,∃ a polynomial $g(X) + h(X)Y$ in C[X,Y] such that :

$$\frac{dq}{d\omega} = \frac{dq}{dX}\cdot\frac{dX}{d\omega} + \frac{dq}{dY}\cdot\frac{dY}{d\omega} = (g(X) + h(X)Y)(\sum_{i=0}^{m} k_i(X)Y^i) = 0$$

(22)

Since $\frac{dq}{d\omega}$ is polynomial in X, Y and $q(X,Y) = 0$ as shown in Equation(21).

Case1: Assuming that m =1 in Equation (21) it implies that:

$$\sum_{i=0}^{1} k_i'(X)Y^i + \sum_{i=0}^{1} ik_i(X)Y^i\left(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2)\right) = (g(X) + h(X)Y)(\sum_{i=0}^{1} k_i(X)Y^i) = 0$$

(23)

The prime means of differential with respect to X .

By matching with the coefficients of Y^i of both sides of (23), we get :

$k_1'(X) = h(X)k_1(X)$ therefore we suppose that $k_1(X)$ is constant and $h(X) = 0$.

$$k_0'(X) = g(X)k_1(X) + h(X)k_0(X)$$

(24)

$$k_1(X)(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2) = g(X)k_0(X)$$

(25)

Taking $k_1(X) = 1$, then balancing the degree of $g(X), k_1(X)$ and $k_0(X)$, we infer to deg($g(X)$) $= 1$ only . Due to this, we suppose that $g(X) = A_1X + A_0$. Then:

$$k_0(X) = \frac{1}{2}A_1X + A_0X + B$$

(26)

Where B is an integration constant.

Substituting $g(X), k_1(X)$ and $k_0(X)$ in Equation(25) , we get :

$$\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2) = (A_1X + A_0) + (\frac{1}{2}A_1X + A_0X + B)$$

(27)

Then, setting all the coefficients of X to zero , we get :

B=0 , $A_0 = \mp\frac{c}{pa^2b}$, $A_1 = \mp\frac{2b(\mu+\gamma)}{3c}$

Substituting A_0 and A_1 in Equation (26), becomes :

$$k_0(X) = \mp\frac{2b(\mu+\gamma)}{3c}X^2 \mp \frac{c}{pa^2b}X$$

(28)

The Equation (21) becomes:

$$0 = \mp\frac{2b(\mu+\gamma)}{3c}X^2 \mp \frac{c}{pa^2b}X + Y$$

(29)

Substituting $Y(\omega)$ in Equation(19) and integrating the equation according to (ω) we obtain :

$$U_1(x,y,t) = \frac{2c^2}{e^{-c\left(\frac{\omega}{a^2bp+3cc_1}\right)-a^2b^2p(\mu+\gamma)}} \quad , \quad W_1(x,y,t) = \frac{b}{a}\frac{2c^2}{e^{-c\left(\frac{\omega}{a^2bp+3cc_1}\right)-a^2b^2p(\mu+\gamma)}}$$

(30)

$$U_2(x,y,t) = \frac{3c^2}{e^{c\left(\frac{\omega}{a^2bp}-3cc_1\right)-2a^2b^2p(\mu+\gamma)}} \quad , \quad W_2(x,y,t) = \frac{b}{a}\frac{3c^2}{e^{c\left(\frac{\omega}{a^2bp}-3cc_1\right)-2a^2b^2p(\mu+\gamma)}}$$

(31)

$$U_{3,4}(x,y,t) = \frac{3c^2}{e^{c\left(\frac{\omega}{a^2bp}-3cc_1\right)+2a^2b^2p(\mu+\gamma)}} \quad , \quad W_{3,4}(x,y,t) = \frac{b}{a}\frac{3c^2}{e^{c\left(\frac{\omega}{a^2bp}-3cc_1\right)+2a^2b^2p(\mu+\gamma)}}$$

(32)

Case2: To improve accuracy, assuming that m =2 in Equation(21) it implies that :

$$\sum_{i=0}^{2} k_i'(X)Y^i + \sum_{i=0}^{2} ik_i(X)Y^i\left(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2)\right) = (g(X) + h(X)Y)(\sum_{i=0}^{2} k_i(X)Y^i) = 0$$

33)

$$k_1(X)(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2) = g(X)k_0(X)$$

(34)

$$k_0'(X) + 2k_2(X)(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2) = g(X)k_1(X) + h(X)k_0(X)$$

(35)

$$k_1'(X) = g(X)k_2(X) + h(X)k_1(X)$$

(36)

$$k_2'(X) = h(X)k_2(X)$$

(37)

Taking $k_2(X) = 1$

$$k_1(X) = \frac{1}{2}A_1X^2 + A_0X + B$$

(38)

$$k_0(X) = \frac{1}{8}{A_1}^2X^4 + \frac{1}{2}A_1A_0X^3 + \frac{1}{2}(A_1B + A_0)X^2 + A_0BX - 2\left(\frac{1}{pa^2b}(cX - b(\mu+\gamma)X^2)\right)X + C$$

(39)

Substituting Equation (38) and Equation (39) into Equation (34) and setting the X coefficients to zero, we get:

$$A_1 = A_0 = B = 0 \quad , \quad k_1(X) = 0$$

(40)

$$k_0(X) = -2\left(\frac{1}{pa^2b}(cX^2 - b(\mu+\gamma)X^3)\right)$$

(41)

$$Y = \pm X\sqrt{\frac{2c}{pa^2b} - \frac{2((\mu+\gamma)}{pa^2}X}$$

(42)

$$X_1 = \frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(c_1+\sqrt{2}\omega)}{2a\sqrt{b}\sqrt{p}})^2}$$

(43)

$$X_2 = \frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(\sqrt{2}\omega-c_1)}{2a\sqrt{b}\sqrt{p}})^2}$$

(44)

$$U_1(x,y,t) = \frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(c_1+\sqrt{2}\omega)}{2a\sqrt{b}\sqrt{p}})^2} \quad , \quad W_1(x,y,t) = \frac{b}{a}\frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(c_1+\sqrt{2}\omega)}{2a\sqrt{b}\sqrt{p}})^2}$$

(45)

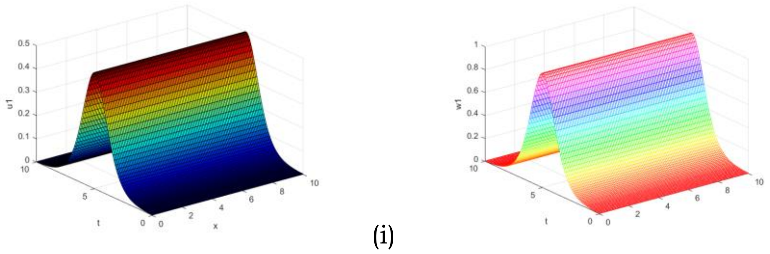
$$U_2(x,y,t) = \frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(\sqrt{2}\omega-c_1)}{2a\sqrt{b}\sqrt{p}})^2} \quad , \quad W_2(x,y,t) = \frac{b}{a}\frac{c}{b(\mu+\gamma)\cosh(\frac{\sqrt{c}(\sqrt{2}\omega-c_1)}{2a\sqrt{b}\sqrt{p}})^2}$$

(46)

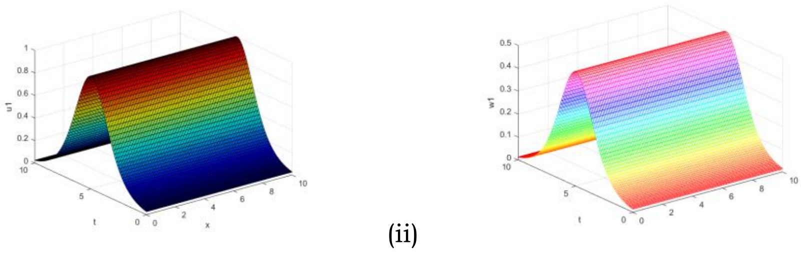
4. Discussion:

Using graphics, this section discusses how significant parameters affect the solution.

Figure 1: obtained two figures (i&ii) for the proposed method solution (U_1 , W_1), when m=2, with a=1,b=2,c=2,c_1=-7, μ=1, γ=1and p=1in Figure(1,i) , with a=2,b=1,c=2,c_1=-7, μ=1, γ =1and p=1in Figure(1,ii) . It is clear through the figures the effect of (b) on the results, where the values of (W_1) multiply while retaining the properties of the wave. As for Figure (ii) it turns out that an increase in the value of (a) corresponds to a decrease in the value of (W_1) the properties and behavior of the physical wave are not affected.



(i)



(ii)

Figure 1. shows the approximate solutions for U_1 and W_1 when m=2:

i) with a=1,b=2,c=2,c_1=-7, μ=1, γ=1and p=1

ii) with a=2,b=1,c=2,c_1=-7, μ=1, γ=1and p=1

Figure 2: In this case, when m=2, with a=1,b=2,c=2,c_1=-7, μ=1, γ=1and p=-10, the effect of the value of (p) on the behavior of the wave was studied, and it was found that the negative value of this parameter indicates a change in the wave behavior and its amplitude.

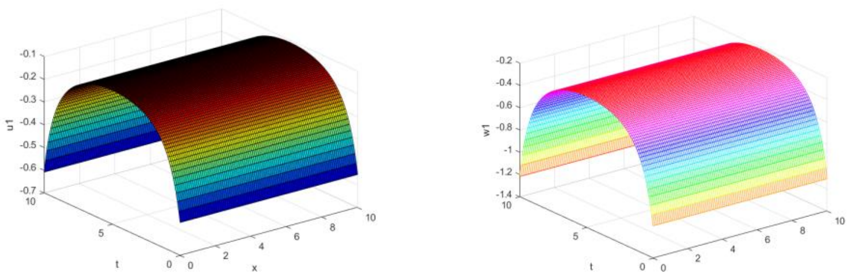


Figure 2. shows the approximate solutions for U_1 and W_1 when m=2, with a=1,b=2,c=2, c_1=-7, μ=1, γ=1and p=-10.

Figure 3: This figure shows the effect of another parameter on the behavior of the wave, when a=1,b=2,c=2,c_1=-10, μ=1, γ=1and p=-10. This parameter is (c_1) and we notice the direction of the value towards the negative leads to a slight displacement of the wave while it almost retains its behavior and characteristics.

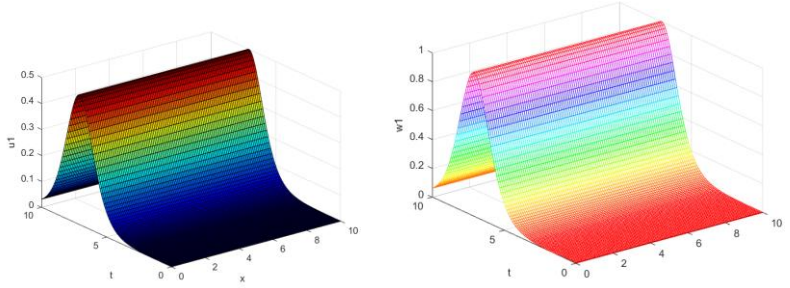
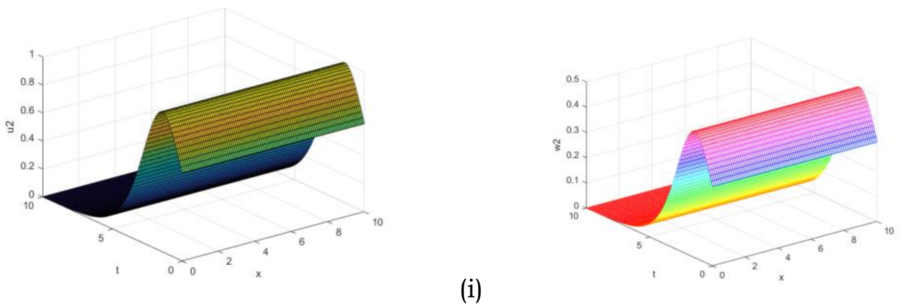
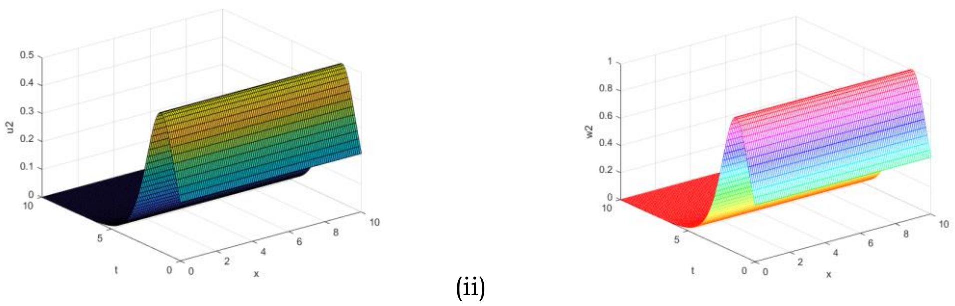


Figure 3. displays the approximate solutions for U_1 and W_1 when m=2, with a=1,b=2,c=2, c_1=-10, μ=1, γ=1and p=-10.

Figure 4: the graphs for the proposed method solution (U_2 , W_2) when m=2, with a=2,b=1,c=2,c_1=2, μ=1, γ=1and p=1 for (i) , and a=1,b=2,c=2,c_1=2, μ=1, γ=1and p=1 for (ii) , It is clear that there is a slight difference between the results compared to the results obtained in Figure (1).



(i)



(ii)

Figure 4. shows the approximate solutions for U_2 and W_2 when m=2:

i) with a=1,b=2,c=2,c_1=-7, μ=1, γ=1and p=1

ii) with a=2,b=1,c=2,c_1=-7, μ=1, γ=1and p=1

Figure 5: The effect of the value of (p) on the behavior of the wave was investigated in this case with a=1,b=2,c=1, c_1=2, μ=1, γ=1 and p=-2, and it was discovered that the negative value of this parameter indicates a change in the wave behavior and its magnitude.

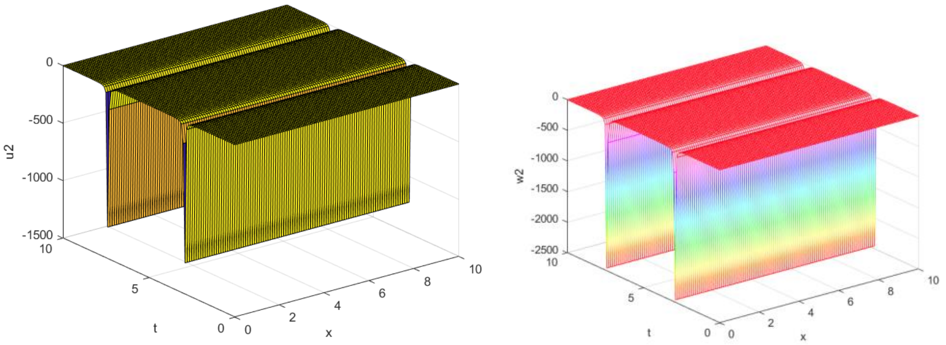


Figure 5. shows the effect of the value of (p) on the behavior of the wave with $a=1, b=2, c=1, c_1=2, \mu=1, \gamma=1$ and $p=-2$.

5. Conclusion

For this study, we utilized a first integral approach to find an approximate solution to the Riemann wave problem. By inputting the equation's parameters, we were able to identify solutions that allowed us to analyze the wave's behavior and features. Through the results and figures, it was found that a change in negative or positive parameters, such as c_1 and p, caused the wave's behavior to change during the specified time period of analysis. We discovered that certain parameters, like a and b, did not impact the wave's behavior. It's worth noting that H.K. Barman successfully applied Kudryashov's method to establish solutions for the RWE, and when comparing the FIM with the method used in [1], we conclude that our approach yielded solutions with fewer steps and higher efficiencies, demonstrating its applicability and effectiveness in solving nonlinear equations.

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