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SJ-FLAT MODULES

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Abstract

In this work we are aiming to explain the notion of SJ-flat Modules. SJ-injectivity notion had been proposed and examined within R-module right M characterised as SJ-N-injective – as N stands for R-module right– while R-homomorphism out of a right submodule for SJ(N)=soc(N) ∩ (J(N)J(R)) within M in an extend till N. Examples showing how SJ-injectivity differs from previous injectivity ideas like nonobjectivity and soc-injectivity are offered to show how this new injectivity condition is linked to existing injectivity notions. Highlighted are a few characteristics of the new injectivity class.

Keywords: SJ-Injective Module; SJ-flat module; FP-Injective Right R-Modules.

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Introduction

In this work we are aiming to explain the notion of SJ-flat Modules. SJ-injectivity notion had been proposed and examined within R-module right M characterised as SJ-N-injective – as N stands for R-module right – while R-homomorphism out of a right submodule for SJ(N)=soc(N) ∩ (J(N)J(R)) within M in an extend till N. Examples showing how SJ-injectivity differs from previous injectivity ideas like nonobjectivity and soc-injectivity are offered to show how this new injectivity condition is linked to existing injectivity notions. Highlighted are a few characteristics of the new injectivity class.

In this article, most R-modules and rings are been connected to their identity. soc (M), J (M), signify, socle of M, Jacobson radical, in order, the class of R-modules right is depicted with $R-\text{Mod}$, as if f.g. is been denoted by finitely generated R-module right . While Jacobson's radical R is denoted by J. The submodule N to M notations.

2. SJ-Flat Modules

As a class of modules related SJ-injective modules, we will introduce SJ-flat modules. Many findings in terms of SJ-injectivity & SJ-flatness are given.

Definition 2.1.1. Let be $M \in R$-Mod. letting M is SJ-flat, if $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$, where $SJ(R_R) = \mathrm{soc}(R_R) \cap (J(R))^2$.

Example 2.1.2. While it is true that all of the flat modules is SJ-flat, the reverse is not correct. As if, $\mathbb{Z}$-module $\mathbb{Z}_n$ no flat anymore $n \geq 2$ (see [2, Example, Page 155]), not obvious enough $\mathbb{Z}_n$ if $\mathbb{Z}$-module is SJ- flat at all $n \geq 2$, since $\mathrm{SJ}(\mathbb{Z}) = 0$.

Theorem 2.1.3. equivalating by the following:

(1) Let M^+ being SJ-injective right R-module.

(2) Let M being SJ-flat left R-module.

(3) $\mathrm{Tor}_1(R/B, M) = 0$, for each f.g.. ideal right B of R, with $B \subseteq SJ(R_R)$.

(4) $\mathrm{Tor}_1(R/A, M) = 0$, for each ideal right A of R,with $A \subseteq SJ(R_R)$.

(5) The sequence $0 \longrightarrow (SJ(R_R))\otimes_R M \longrightarrow R_R \otimes_R M$ is exact.

(6) The sequence $0 \longrightarrow A\otimes_R M \longrightarrow R_R \otimes_R M$ is exact for each f.g. ideal right A with R , with $A \subseteq SJ(R_R)$.

Proof. (1)⇒(2) Letting M been an SJ-flat left R-modules, and so on $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$. Using Theorem 1.2.18 in [7], we get $\mathrm{Ext}^1(R/(SJ(R_R)), M^+) \cong (\mathrm{Tor}_1(R/(SJ(R_R)), M))^+$.Since $(\mathrm{Tor}_1(R/(SJ(R_R)),M))^+ = 0$, it follows that $\mathrm{Ext}^1(R/(SJ(R_R)), M^+) = 0$. And so on M^+ is a right SJ-injective R-modules, using Proposition 2.1.8 in [7].

(2)⇒(1) Let M^+ is an SJ-injective right R-modules, and so on $\mathrm{Ext}^1(R/(SJ(R_R)), M^+) = 0$, by [7, Proposition 2.1.8]. By Theorem 1.2.18 in [7], $\mathrm{Ext}^1(R/(SJ(R_R)), M^+) \cong (\mathrm{Tor}_1(R/(SJ(R_R)), M))^+$ & hence $(\mathrm{Tor}_1(R/(SJ(R_R)), M))^+ = 0$. And so on $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$ & hence M is an SJ-flat modules.

(2)⇒(3) Letting A be any ideal right of R, with $A \subseteq SJ(R_R)$. By [4, Theorem 3.2.1, P.75] & Proposition 2.1.8, $\mathrm{Tor}_1(R/A, M)^+ \cong \mathrm{Ext}^1(R/A, M^+) = 0$. And so on $\mathrm{Tor}_1(R/A, M) = 0$, since $\mathbb{Q}/\mathbb{Z}$ is injective cogenerator.

(3)⇒(1) is been clear.

(4)⇒(3) N is a ideal right of R, as: $N \subseteq SJ(R_R)$, so $N = \varinjlim N_i$, where N_i is an f.g. ideal right of R, with $N_i \subseteq SJ(R_R)$ for all i. Let $f_{ij}: N_i \longrightarrow N_j$ (with $i \leq j$) is the inclusion map, & (N_i, f_{ij}) as direct system (see [4, Example 1.5.5 (2), p. 32]). Obviously, $(R/N_i, h_{ij})$ is a R-

modules direct system where $h_{ij}: R/N_i \longrightarrow R/N_j$ is depicted by $h_{ij}(a+N_i) = a + N_j$ with direct limit $(h_i\,, \varinjlim R/N_i)$. The following diagram is commutative:

$$
\begin{tikzcd}
0 \arrow[r] & N_i \arrow[r, "i_i"] \arrow[d, "f_{ij}"] & R \arrow[r, "\pi_i"] \arrow[d, equal] & R/N_i \arrow[r] \arrow[d, "h_{ij}"] & 0 \\
0 \arrow[r] & N_j \arrow[r, "i_j"] & R \arrow[r, "\pi_j"] & R/N_j \arrow[r] & 0
\end{tikzcd}
$$

where i_i & π_i are the inclusion & canonical maps, respectively. By [14, 24.6, p.200], We get the required sequence as: $0 \longrightarrow N \xrightarrow{i} R \xrightarrow{u} \varinjlim R/N_i \longrightarrow 0$. following [14, 24.4, p.199], we get the commutativity of the next diagram:

$$
\begin{tikzcd}
R \arrow[r, "\pi_i"] \arrow[d, equal] & R/N_i \arrow[r] \arrow[d, "h_i"'] & 0 \\
R \arrow[r, "u"] & \varinjlim R/N_i \arrow[r] & 0
\end{tikzcd}
$$

So the family of mappings $\left\{g_i \,|\, g_i: R/N_i \longrightarrow R/\varinjlim N_i \text{ , while } g_i(a+N_i) = a + \varinjlim N_i\right\}$ obtaining homomorphisms direct system, since for $i \leq j$, having $g_j h_{ij}(a+N_i) = g_j\big(a+N_j\big) = a + \varinjlim N_i = g_i(a+N_i)$ for each $a + N_i \in R/N_i$. Hence, we get diagram of R-homomorphism α as the following:

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & R \arrow[r, "\pi_i"] \arrow[d, equal] \arrow[rr, bend left, "u"] & R/N_i \arrow[r, "h_i"] \arrow[d, equal] & \varinjlim R/N_i \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & I \arrow[r, "i"] & R \arrow[r, "\pi_i"] \arrow[rr, bend right, "\pi"'] & R/N_i \arrow[r, "g_i"] & R/\varinjlim N_i \arrow[r] & 0
\end{tikzcd}
$$

with rows are short exact (see [14, p.197]), when π is a canonical map, so could it follows from [1, Exercise 11 (1), p. 52] that $\varinjlim R/N_i \cong R/\varinjlim N_i$. as,

$\mathrm{Tor}_1(R/N, M) = \mathrm{Tor}_1\left(R/\varinjlim N_i\,, M\right)$

$\cong \mathrm{Tor}_1\left(\varinjlim R/N_i\,, M\right)$ (by [5, Theorem XII.5.4 (4), p. 494]).

$\cong \varinjlim \mathrm{Tor}_1(R/N_i\,, M) = 0$ by [10, Proposition 7.8, p. 410]). That means (4)⇒(3) is satisfied.

(3)⇒(4) is been obvious.

(1)⇒(5) letting M be an SJ-flat left R-modules, and so on $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$. Taking required sequence $0 \longrightarrow (SJ(R_R)) \xrightarrow{i} R \xrightarrow{\pi} R/(SJ(R_R)) \longrightarrow 0$. By [5, Theorem X.II.5.4(3),

p. 494], getting the required sequence $0 \longrightarrow \mathrm{Tor}_1(R/(SJ(R_R)), M) \longrightarrow (SJ(R_R)) \otimes_R M \longrightarrow R_R \otimes_R M$. Since $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$, getting the sequence $0 \longrightarrow (SJ(R_R)) \otimes_R M \longrightarrow R_R \otimes_R M$ is exact .

(5)⇒(1) Suppose that the sequence $0 \longrightarrow (SJ(R_R)) \otimes_R M \longrightarrow R_R \otimes_R M$ is exact. By [5, Theorem XII.5.4(3), p. 494], the sequence $0 \longrightarrow \mathrm{Tor}_1(R/(SJ(R_R)), M) \longrightarrow (SJ(R_R)) \otimes_R M \longrightarrow R_R \otimes_R M$ is exact. And so on $\mathrm{Tor}_1(R/(SJ(R_R)), M) = 0$ & hence M be an SJ-flat left R-modules.

(4)⟺(6) By similar way of (1)⟺(5).□

Note: We will use the symbol SJI (resp. SJF) to denote the class of right SJ-injective – Left SJ-flat – R-modules.

Corollary 2.1.4. Both (SJF, SJI) is an almost dual pair.

Proof. Following from Theorem 3.1.3 & Theorem 2.1.3 ((1) & (5)see [7]).

Proposition 2.1.5. as R ring, the next are true:

(1) $SJ(R_R)$ been f.g, so the following pure form of submodule always of SJ-injective R-module right being SJ-injective.

(2) all SJ-flat pure form of submodule of R-module left is SJ-flat.

(3) all direct limits (direct sums) of SJ-flat R-modules left being SJ-flat.

(4) Unless M, N are left R-modules, $M \cong N$, and M is SJ-flat, so N is SJ-flat.

Proof. (1) Letting M be an SJ-injective right R-module & N be a pure form of submodule of M. To prove that N is SJ-injective. Since $R/SJ(R_R)$ is a finitely presented, the sequence $\mathrm{Hom}_R(R/SJ(R_R), M) \longrightarrow \mathrm{Hom}_R(R/SJ(R_R), M/N) \longrightarrow 0$ is the required. By [5, Theorem XII.4.4 (4), p. 491], we get the required sequence $\mathrm{Hom}_R(R/SJ(R_R), M) \longrightarrow \mathrm{Hom}_R(R/SJ(R_R), M/N) \longrightarrow \mathrm{Ext}^1(R/SJ(R_R), N) \longrightarrow \mathrm{Ext}^1(R/SJ(R_R), M)$

, it follows that $\mathrm{Ext}^1(R/SJ(R_R), N) = 0$. By [7, Proposition 2.1.8], N is an SJ-injective R-module right.

(2), (3) & (4), are following from[7, Corollary 2.1.4]& [9, Proposition 4.2.8, p. 70]. □

Corollary 2.1.6. A ideal right $SJ(R_R)$ of a ring R is f.g. iff all of the FP-injective right R-module is SJ-injective.

Proof. (⇒) Let M be an FP-injective right R-module, and so on $\mathrm{Ext}^1(N, M) = 0$, for all finitely presented right R-modules N. By hypothesis, $SJ(R_R)$ is f.g. & hence $R/SJ(R_R)$ is finitely presented, so $\mathrm{Ext}^1(R/SJ(R_R), M) = 0$. Into Proposition 2.1.8 in [7], M is SJ-injective.

(⇐) Let M be an FP-injective R-modules right & let $f: SJ(R_R) \longrightarrow M$ be a right R-homomorphism. By hypothesis, M is SJ-injective, so f continues to R. Within[7, Proposition 1.2.33], $SJ(R_R)$ is f.g. □

Theorem 2.1.7. Equivalent to the following of ring R:

(1) f.g. ideal right I for R with $I \subseteq SJ(R_R)$ is finitely presented always.

(2) Letting M be a right SJ-injective R-module, so M^{++} is SJ-injective.

(3) Letting M be a right SJ-injective R-module, so M^{+} is SJ-flat.

(4) A left R-module N is SJ-flat iff N^{++} is SJ-flat.

(5) within direct products SJF is closed.

(6) ${}_R R^S$ is SJ-flat for index set S at all.

(7) $\mathrm{Ext}^2(R/I, M) = 0$ for each FP-injective R-module M right & always f.g. ideal right I of R by $I \subseteq SJ(R_R)$.

(8) If $0 \longrightarrow N \longrightarrow M \longrightarrow H \longrightarrow 0$ being a required sequence of right R-modules with N is FP-injective & M is SJ-injective, while $\mathrm{Ext}^1(R/I, H) = 0$, to all f.g. ideal right I of R with $I \subseteq SJ(R_R)$.

(9) R-module left has an (SJF)-pre-envelope at all.

Proof.(1)⇒(2) Suppose that M is SJ-injective. Let I be an f.g. ideal right of R with $I \subseteq SJ(R_R)$. And so on $\mathrm{Ext}^1(R/I, M) = 0$. Within hypothesis, I finitely presented & hence is a required sequence $B_2 \xrightarrow{f_2} B_1 \xrightarrow{f_1} I \longrightarrow 0$ in which B_i is an f.g. free right R-module, $i = 1,2$. And so on getting the sequence $B_2 \xrightarrow{f_2} B_1 \xrightarrow{\alpha} R \xrightarrow{\pi} R/I \longrightarrow 0$ is the required, where $\alpha = if_1$ & $i: I \longrightarrow R$ & $\pi: R \longrightarrow R/I$ including & canonical homomorphisms, respectively. And so on R/I is 2- presented & so on Lemma 1.2.24(see[7]) implies that $\mathrm{Tor}_1(R/I, M^+) \cong (\mathrm{Ext}^1(R/I, M))^+ = 0$. So, M^+ is an SJ-flat R-module left .

(2)⇒(3) Let M be SJ-injective. Within hypothesis, M^+ is an SJ-flat left R-module. By Theorem 2.1.3, M^{++} is SJ-injective.

(3)⇒(4) Suppose that N is an SJ-flat left R-module, and so on Theorem 2.1.3 implies that, N^+ being an SJ-injective right R-module. By (3), N^{+++} is SJ-injective. By Theorem 2.1.3, N^{++} is SJ-flat. Conversely, suppose that N^{++} is SJ-flat. Through Theorem 1.2.30(1)(see [7]), N is N^{++} as a pure submodule. Within Proposition 2.1.5(2), N is SJ-flat.

(4)⇒(5) By hypothesis, $(SJF)^{++} \subseteq SJF$. Since (SJF , SJI) is an mainly dual pair (by Corollary 2.1.4), getting from [9, Proposition 2.3.1 & Proposition 2.2.8 (3), p.85 & 70], that SJF is definable. So SJF is closed with direct products.

(5)⇒(6) obvious.

(6)⇒(1) Letting K be an f.g. ideal right of R with $K \subseteq SJ(R_R)$. Within (6), $\prod R = R^I$ is an SJ-flat left R-module. And so on $\mathrm{Tor}_1(R/K, \prod R) = 0$, by Theorem 2.1.3. And so on, getting the exact row by commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K\otimes_R(\prod R) \arrow[r] \arrow[d, "\alpha"'] & R\otimes_R(\prod R) \arrow[r] \arrow[d, "\beta"'] & R/K\otimes_R(\prod R) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \prod K \arrow[r] & \prod R \arrow[r] & \prod R/K \arrow[r] & 0
\end{tikzcd}$$

Using [4, Theorem 3.2.22, p.81], we get γ & β are isomorphisms. By [8, Lemma 3.2,p. 14], α is an isomorphism. Using [4, Theorem 3.2.22, p. 81], K is finitely presented.

(1)⇒(7) Let I being an f.g. ideal right of R for $I \subseteq SJ(R_R)$ & as M been an FP-injective R -module right. By (1), I is finitely presented. By [5, Theorem XII.4.4 (3), p. 491], by getting required sequence $\mathrm{Ext}^1(I, M) \longrightarrow \mathrm{Ext}^2(R/I, M) \longrightarrow \mathrm{Ext}^2(R, M)$. But $\mathrm{Ext}^1(I, M) = 0$ (since M is FP-injective & I is finitely presented) & $\mathrm{Ext}^2(R, M) = 0$ (as R is projective). And so on $\mathrm{Ext}^2(R/I, M) = 0$.

(7)⇒(8) If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an required sequence of right R-modules, where N is FP-injective & M is SJ-injective & let I be an f.g. ideal right of R with $I \subseteq SJ(R_R)$. By [5, Theorem XII.4.4 (4), p.491], We get an required sequence $0 = \mathrm{Ext}^1(R/I, M) \longrightarrow \mathrm{Ext}^1(R/I, K) \longrightarrow \mathrm{Ext}^2(R/I, N) = 0$. And so on $\mathrm{Ext}^1(R/I, K) = 0$ for each f.g. ideal right I of R with $I \subseteq SJ(R_R)$.

(8)⇒(1) Let N be an FP-injective right R-module, and so on getting the required sequence $0 \longrightarrow N \longrightarrow E(N) \longrightarrow E(N)/N \longrightarrow 0$, where $E(N)$ is the injective envelope of N. Let I be an f.g. ideal right of R with $I \subseteq SJ(R_R)$. By hypothesis, $\mathrm{Ext}^1(R/I, E(N)/N) = 0$. So it follows from [5, Theorem XII.4.4 (4), p. 491] that the sequence $0 = \mathrm{Ext}^1(R/I, E(N)/N) \longrightarrow \mathrm{Ext}^2(R/I, N) \longrightarrow \mathrm{Ext}^2(R/I, E(N)) = 0$ is exact, & so

$\mathrm{Ext}^2(R/I, N) = 0$. Since $0 \to I \to R \longrightarrow R/I \longrightarrow 0$ is exact, so it is follows from [5, Theorem XII.4.4 (3), p. 491] that the sequence $0 = \mathrm{Ext}^1(R, N) \longrightarrow \mathrm{Ext}^1(I, N) \longrightarrow \mathrm{Ext}^2(R/I, N) = 0$. And so on $\mathrm{Ext}^1(I, N) = 0$ & this implies that I is finitely presented by[7, Remark 1.2.34].

(5)⇔(9) By Corollary 2.1.4, the pair (SJF, SJI) is an almost dual pair. By [9, Proposition 4.2.8 (3), p. 70], SJF is close with direct products iff all of the left R-module has an SJF-pre-envelope.□

Theorem 2.1.8. The following are equivalent for a ring R:

(1) All of the f.g. ideal right I of R with $I \subseteq SJ(R_R)$ is finitely presented & the class SJI is with pure submodules being closed.

(2) A right R-module M is SJ-injective iff M^+ is SJ-flat.

(3) $SJ(R_R)$ is f.g. & all of the pure quotient of SJ-injective right R-module is SJ-injective

(4) SJI is closed with direct limits.

(5) The next conditions explain:

(a) All of the right R-module has an (SJI)-cover.

(b) All of the pure quotient of SJ-injective R-module right is SJ-injective.

Proof. (1)⇒(2) Let M^+ be SJ-flat, so M^{++} is SJ-injective by Theorem 2.1.3. Since M is a pure form of submodule of M^{++} (by Theorem 1.2.30(1) in [7]), it follows from hypothesis, that M is SJ-injective. The converse by Theorem 2.1.7.

(2)⇒(1) Let $M \in SJI$ & let N be a pure form of submodule of M. By hypothesis, $(SJI)^+ \subseteq SJF$. Since (SJF, SJI) is an almost dual pair (by Corollary 2.1.4), we get from [9,Theorem 4.3.2] that $N^+ \in SJF$. By hypothesis, $N \in SJI$. And so on the class SJI is with pure submodules being closed. Let I be any f.g. ideal right of R with $I \subseteq SJ(R_R)$. By hypothesis, $(SJI)^+ \subseteq SJF$. By Theorem 2.1.7, I is a finitely presented ideal.

(2)⇒(3) Let $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ be a pure required sequence of right R-modules with M is SJ-injective, so it follows from Theorem 1.2.27 in [7] that the sequence $0 \longrightarrow (M/N)^+ \longrightarrow M^+ \longrightarrow N^+ \longrightarrow 0$ is split. By hypothesis, M^+ is SJ-flat, so $(M/N)^+$ is SJ-flat. And so on M/N is SJ-injective by hypothesis one more time. Now, we will prove that $SJ(R_R)$ is an f.g. right R-module. Let $\{M_i\}_{i\in I}$ be a family of SJ-injective right R-modules. By Theorem 2.1.3(1), $\prod_{i\in I} M_i$ is SJ-injective. By [9, Lemma 2.2.6(1), p.24], $\bigoplus_{i\in I} M_i$ is a pure form of submodule of $\prod_{i\in I} M_i$. Since (1) & (2) are equivalent, getting that the class SJI is with pure submodules being closed. Since $\prod_{i\in I} M_i \in SJI$, getting that $\bigoplus_{i\in I} M_i$ is SJ-injective. By Corollary 2.1.24 in [7], $SJ(R_R)$ is an f.g. right R-module.

(3)⇒(4) Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a direct system of SJ-injective right R-modules. By [14, 33.9 (2), p. 279], it will bea pure required sequence $\bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow \varinjlim M_\alpha \longrightarrow 0$. Since $SJ(R_R)$ is an f.g. right R-module, $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is SJ-injective (by Corollary 2.1.24). And so on $\varinjlim M_\alpha$ is SJ-injective by hypothesis.

(4)⇒(2) Let $\{K_i : i \in I\}$ be a family of injective right R-modules. Since $\bigoplus_{i\in I} K_i = \varinjlim \{\bigoplus_{i\in I_0} K_i : I_0 \subseteq I, I_0 \text{ finite } \}$ (see [14, p. 206]), so $\bigoplus_{i\in I} K_i$ is SJ-injective. By Corollary 2.1.24, $SJ(R_R)$ is an f.g. right R-module. By Proposition 2.1.5 (1), SJI is with pure submodules being closed. Since SJI is closed with direct products (by Theorem 2.1.3 (1)) & since SJI is closed with direct limits (by hypothesis), getting SJI is a definable class. (SJI, SJF) is an almost dual pair & hence a right R-modules M is SJ-injective iff M^+ is SJ-flat.

(3)⇔(5) By Corollary 2.1.24 & Theorem 1.2.29 (see [7]). □

Corollary 2.1.9. Let R be a ring. If all of the f.g. ideal right I of R with $I \subseteq SJ(R_R)$ is finitely presented, so the following statements are equivalent:

(1) All of the SJ-flat left R-module is flat.

(2) All of the SJ-injective right R-module is FP-injective.

(3) All of the SJ-injective pure injective right R-module is injective.

Proof. (1)⇒(2) Let M be an SJ-injective right R-module, so M^+ is SJ-flat (by Theorem 2.1.7), & so M^+ is flat by hypothesis. And so on M^{++} is injective by Proposition 1.2.11(see [7]). Since M is a pure form of submodule of M^{++}, it follows that M is an FP-injective module.

(2)⇒(3) Let M be an SJ-injective pure injective right R-module. By (2), M is FP-injective. By [11, Proposition 2.6 (c), p. 324], all of the exact sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ is pure. Since M is pure injective (by hypothesis), it is follows from[7, Theorem 1.2.28], that all of the pure sequence $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ is split. And so on all of the required sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ is split. By Theorem 1.2.5 in [7], M is injective.

(3)⇒(1) Suppose that N is an SJ-flat left R-module, and so on N^+ is an SJ-injective pure injective by [7,Theorem 2.1.3 & Theorem 1.2.30(2)].And so on N^+ is injective & so N is flat by Proposition 1.2.11 (see [7])□

Proposition 2.1.10. If SJI is closed with direct limits, so equivalating by the next:

(1) R is a right SJ-injective ring.

(2) All of the left R-module has a SJ-flat monic pre-envelope.

(3) All of the flat right R-module is SJ-injective.

(4) All of the injective left R-module is SJ-flat.

Proof. (1)⇒(2) Suppose N be a R-module left, so it will be an epimorphism $\alpha: R_R^{(S)} \longrightarrow N^+$ as an set of index S (with [10, Theorem 2.35, p. 58]). with applying [5, Proposition XI.2.3, p. 420], $\alpha^+: N^{++} \longrightarrow (R^{(S)})^+$ is an R-monomorphism. By [14, 11.10 (2) (ii), p. 87], $(R^{(S)})^+ \cong (R_R^+)^S$. By[7, Theorem 1.2.30 (1)], N is a pure form of submodule of N^{++}. So it will be an R-monomorphism $g: N \longrightarrow (R_R^+)^S$. Since SJF is closed with direct products (by hypothesis & Theorem 2.1.7), it follows from Theorem 2.1.7 that N has an (SJF)-pre-envelope $f: N \longrightarrow F$ & $(R_R^+)^S$ is SJ-flat. And so on it will be an R-homomorphism $h: F \longrightarrow (R_R^+)^S$ like $hf = g$, so this means that f is an R-monomorphism.

(2)⇒(3) Let N be an injective left R-module. By hypothesis, it will be an R-monomorphism $f: N \longrightarrow F$ with F is SJ-flat. But $N \cong f(N) \subseteq^{\oplus} F$, so getting that N is SJ-flat by Proposition 2.1.5 (4).

(3)⇒(4) Let M be a flat right R-module, so M^+ is injective, and so then it is SJ-flat (by hypothesis). By Theorem 2.1.3, M^{++} is SJ-injective. Since M is a pure form of submodule of M^{++} (by Theorem 1.2.30 (1)in [7]), it applying from Theorem 2.1.8 that M is SJ-injective.

(4)⇒(1) Obvious, since R_R is flat. □

Proposition 2.1.11. The class SJF is closed with homomorphisms kernels if $\ker(\alpha)$ is SJ-flat, for each SJ-flat R-module M left and $\alpha \in \mathrm{End}(M)$.

Proof. (⇒) Obvious.

(⇐) Suppose that $f: N \longrightarrow M$ is any R-homomorphism with N & M are SJ-flat left R-modules. Define $\alpha: N \oplus M \longrightarrow N \oplus M$ by $\alpha\big((x,y)\big) = \big(0, f(x)\big)$. So $\ker(\alpha) = \ker(f) \oplus M$ is SJ-flat by hypothesis & hence $\ker(f)$ is SJ-flat. □

Theorem 2.1.12. If R is a commutative ring, so the following statements are equivalent:

(1) All of the f.g. ideal right I of R with $I \subseteq SJ(R_R)$ is finitely presented .

(2) $\mathrm{Hom}_R(M,N)$ is SJ-flat as every SJ-injective R-modules M & every injective R-modules N.

(3)$\mathrm{Hom}_R(M,N)$ is SJ-flat as every projective R-modules M & N.

(4) $\mathrm{Hom}_R(M,N)$ is SJ-flat as every injective R-modules M & N.

(5) $\mathrm{Hom}_R(M,N)$ is SJ-flat as every projective R-modules M & every SJ-flat R-modules N.

Proof. (1)⇒(2) Supposing M been an SJ-injective R-module & N an injective R-module. Let K be an f.g. ideal right of R with $K \subseteq SJ(R_R)$, so K is finitely presented. By [5, Theorem XII.4.4(3), p.491], getting the required sequence $0 \longrightarrow \mathrm{Hom}_R(R/K, M) \longrightarrow \mathrm{Hom}_R(R,M) \longrightarrow \mathrm{Hom}_R(K,M) \longrightarrow 0$. And so, on the sequence $0 \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K,M),N) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(R,M),N) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(R/K, M),N) \longrightarrow 0$ is exact by [5, Theorem XII.4.4 (3), p. 491] one more time. There as, getting the required sequence $0 \longrightarrow \mathrm{Hom}_R(M,N)\otimes_R K \longrightarrow \mathrm{Hom}_R(M,N)\otimes_R R \longrightarrow \mathrm{Hom}_R(M,N)\otimes_R (R/K) \longrightarrow 0$ by [4, Theorem 2.2.11, p. 78] & hence $\mathrm{Hom}_R(M,N)$ is SJ-flat.

(2)⇒(3) obvious.

(3)⇒(1) By [2, Proposition 2.3.4, p. 66] & [10, Theorem 2.75, p.92], getting that $(R^{++})^S \cong \left(\mathrm{Hom}_{\mathbb{Z}}(R^+\otimes_R R, \mathbb{Q}/\mathbb{Z})\right)^S \cong \left(\mathrm{Hom}_R(R^+,R^+)\right)^S$ as any index set S. There as, $(R^{++})^S \cong \mathrm{Hom}_R(R^+,(R^+)^S)$ is SJ-flat as any index set S by [14, 11.10 (2), p. 87] and since R^+ & $(R^+)^S$ are injective. Since R^S is a pure form of submodule of $(R^{++})^S$ by [7,Theorem 1.2.30] & [3, Lemma 1 (2)], it follows that R^S is SJ-flat as any index set S by Proposition 2.1.5 (2). And so on (1) is satisfied by Theorem 2.1.7.

(1)⇒(5) Let M be a projective right R-module & N be SJ-flat, and so on it will be a projective right R-module P with $M \oplus P \cong R^{(S)}$ as some index set S. There as, $\mathrm{Hom}_R(M,N) \oplus \mathrm{Hom}_R(P,N) \cong \mathrm{Hom}_R(R^{(S)},N) \cong \left(\mathrm{Hom}_R(R,N)\right)^S \cong N^S$ by [14, 11.10 & 11.11, p. 87 & 88]. By Theorem 2.1.7, N^S is SJ-flat, and so $\mathrm{Hom}_R(M,N)$ is SJ-flat.

(5)⇒(4) Obvious.

(4)⇒(1) By [14, 11.10 & 11.11, p. 87 & 88], getting that $R^S \cong \mathrm{Hom}_R(R^{(S)},R)$ as any index set S. By (4), R^S is SJ-flat, and so on Theorem 2.1.7 implies that every of the f.g. ideal right I of R with $I \subseteq SJ(R_R)$ is finitely presented . □

Corollary 2.1.13. Suppose R as a commutative ring with SJI is closed with direct products. So the following are equivalent:

(1) As M been an SJ-injective R-module.

(2) $\mathrm{Hom}_R(M,N)$ is SJ-flat, as injective R-module N.

(3) $M\otimes_R N$ is SJ-injective, as flat R-module N.

Proof. (1)⇒(2) Applying Theorem 2.1.12.

(2)⇒(3) By [10, Theorem 2.75, p. 92], getting that $(M\otimes_R N)^+ \cong \mathrm{Hom}_R(M,N^+)$ as any R-module N. If N is flat, so N^+ is injective, and so on $(M\otimes_R N)^+$ is hypothesis by SJ-flat. Hence $M\otimes_R N$ is SJ-injective by Theorem 2.1.8.

(3)⇒(1) It follows from [2, Proposition 2.3.4, p. 66], since R is a flat. □

Corollary 2.1.14. Let R be a commutative ring with SJF is closed with homomorphisms kernels & direct products. So, the following statements hold as any R-module N:

(1) $\mathrm{Hom}_R(M,N)$ is SJ-flat, as any SJ-injective R-module M.

(2) $\mathrm{Hom}_R(N,M)$ is SJ-flat, as any SJ-flat R-module M.

(3) $M\otimes_R N$ is SJ-injective, as any SJ-injective R-module M.

Proof. (1) Suppose that M is an SJ-injective R-module. Clearly, the required sequence $0 \longrightarrow N \longrightarrow E_0 \longrightarrow E_1 \longrightarrow 0$ induces the required sequence $0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,E_0) \longrightarrow \mathrm{Hom}_R(M,E_1) \longrightarrow 0$ where E_0 & E_1 are injective R-modules. And so on $\mathrm{Hom}_R(M,E_0)$ & $\mathrm{Hom}_R(M,E_1)$ are SJ-flat by Theorem 2.1.12 (2). Hence $\mathrm{Hom}_R(M,N)$ is SJ-flat (by hypothesis).

(2) Suppose that M is an SJ-flat R-module, so we get the required sequence $0 \longrightarrow \mathrm{Hom}_R(N,M) \longrightarrow \mathrm{Hom}_R(F_0,M) \longrightarrow \mathrm{Hom}_R(F_1,M) \longrightarrow 0$ where F_0 & F_1 are free R-modules, so F_0 & F_1 are projective R-modules by [2, Proposition 5.2.6., p. 146] .And so on the modules $\mathrm{Hom}_R(F_0,M)$ & $\mathrm{Hom}_R(F_1,M)$ are SJ-flat by Theorem 3.1.12. There as, $\mathrm{Hom}_R(N,M)$ is SJ-flat (by hypothesis).

(3) Suppose that M is any SJ-injective R-module, so getting that $(M\otimes_R N)^+ \cong \mathrm{Hom}_R(M,N^+)$ is SJ-flat by [10, Theorem 2.75, p.92] & applying (1). By Theorem 2.1.8, $M\otimes_R N$ is SJ-injective. □

Proposition 2.1.15. Suppose R be a commutative ring. So the next equivalenting the findings:

(1) M is an SJ-flat R-module.

(2) $\mathrm{Hom}_R(M,N)$ is SJ-injective, as every injective R-module N.

(3) $M\otimes_R N$ is SJ-flat, as every flat R-module N.

Proof. (1)⇒(2) Suppose that N is any injective R-module. By [4, Theorem 3.2.1, p. 75], $\mathrm{Ext}^1\big(R/(SJ(R_R))\ ,\mathrm{Hom}_R(M,N)\big) \cong (\mathrm{Hom}_R(\mathrm{Tor}_1(R/(SJ(R_R))\ ,M),N) = 0$. By Theorem 3.1.8, $\mathrm{Hom}_R(M,N)$ is SJ-injective.

(2)⇒(3) Suppose that N is a flat R-module, so N^+ is injective by [10, Proposition 3.54, p.136]. Since $(M\otimes_R N)^+ \cong \mathrm{Hom}_R(M,N^+)$ by [10, Theorem 2.75, p. 92], & $\mathrm{Hom}_R(M,N^+)$ is SJ-injective by hypothesis, so $(M\otimes_R N)^+$is SJ-injective. Hence $M\otimes_R N$ is SJ-flat by Theorem 2.1.3.

(3)⇒(1) This follows from [2, Proposition 2.3.4, p. 66]. □

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