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INITIAL VALUE PROBLEMS OF LINEAR HEAT EQUATION SOLVABLE BY USING CLASSICAL SYMMETRIES

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Abstract

In this paper, we investigated the use of classical symmetries to solve initial value problems(IVPs) for the linear heat equation w_t= w_xx. In order to slove a problem of type IVP s ,it is relied that we have to make the given condition w(x,t) |_(t=t_0=0)=G(x) need be left invariant under the one parameter Lie group of transformation that leaves the linear heat equation invariant. Under this procedure, the some initial conditions solvable by using classical symmetries, were discussed and several examples were then given.

Keywords: Classical Symmetries, Linear Heat Equation, Lie Symmetry Analysis, Initial Value Problems, Left Invariant.

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1. Introduction

The most widely used application method for constructing exact solutions for partial differential equations(PDEs),that is symmetry analysis [3,4,8,9,19,21]. It has not been successful in the treatment of (IVPs) [20]. In practice, the initial conditions have been just as important as the governing equations for determining the behavior of a system[18,23]. There are many papers have been dedicated to investagte initial value problems by using symmetry group analysis such as classical symmetry [2,12,17,26],nonclassical symmetry[2,12,15,26] ,higher conditional symmetry[6,10,16],etc.
At present, it must be satistied three criteria to be accepted view is that a symmetry of IVPs [5,20,25,26]. The criteria are that:

(a)A symmetry of the governing differential equations.
(b)A smooth bijective mapping of the domain to itself.
(c)A mapping of the set of initial data to itself
.

The formal definition of a symmetry of an initial value problems (IVPs) are given by the above three criteria [5,25].
Consider the following linear heat equation in one dependent variable (w) and two independent variables $x,\ t$ (see e.g [5])

$$w_t = w_{xx}$$

with initial conditions

$$w(x,t)|_{t=t_0=0} = G(x)$$

The linear heat equation is experienced in many fields of engineering, mathematics, physics, chemistry and biology. Also, the linear heat equation is extensively used to describe the complex physical phenomena in various fields of science and engineering, mainly in fluid mechanics, solid state physics and chemical physics etc. [1,7,11,13,14,22].

2.Initial Value Problems Solvable by Classical Symmetries:

In this section, we consider that the initial value problem(IVP) of the governing equation

$$w_t = w_{xx} \tag{2.1}$$

With the intial condition

$$w(x,t)|_{t=t_0} = G(x) \tag{2.2}$$

Admits the symmetry generator

$$\Gamma = \delta(x,t,w)\frac{\partial}{\partial x} + \tau(x,t,w)\frac{\partial}{\partial t} + \omega(x,t,w)\frac{\partial}{\partial w} \tag{2.3}$$

Which leaves heat eqution(2.1) ivariant. According to equation(2.1) and the infinitesimal operator (2.3), the invariant surface condition(ISc) is

$$\delta(x,t,w)w_x + \tau(x,t,w)w_t = \omega(x,t,w).$$

Intial conditions(2.2) that are invariant under (2.3), satisfy the conditions

$$\Gamma(t-t_0)|_{t=t_0} = 0, \tag{2.4}$$

With

$$\Gamma(w-G(x))|_{w=G(x),t=t_0} = 0. \tag{2.5}$$

Now, we will find some general intial condition $w(x,0) = G(x)$ admitted by using classical symmetries for the linear heat equation (2.1).
In [24]. Olver P.J studied classical symmetries of the linear heat equation that we will be used in this paper.

2.1 The (IVPs) of linear Heat Equation Solvable by Classical Symmetries:

We now suppose that the initial value problem composed of the goverening equation that is linear heat equation by using the classical symmetries.
2.1.1 Initial Condition Left Invariant
The PDE(2.1)

$$w_t = w_{xx},$$

Has a classical symmetry with finite- dimensional generator[24]

$$\Gamma = [c_1 + c_4x + 2c_5t + 4c_6xt]\frac{\partial}{\partial x} + [c_2 + 2c_4t + 4c_6t^2]\frac{\partial}{\partial t} + [c_3 - c_5x - c_6x^2 - 2c_6t]w\frac{\partial}{\partial w}, \tag{2.6}$$

From (2.4), we have

$$\Gamma(t-t_0)|_{t=t_0} = 0 \to c_2 = 0.$$

From (2.5), we get

$$G'(x)[c_1 + c_4 x] + G(x)[c_6 x^2 + c_5 x - c_3] = 0, \quad (2.7)$$

By solving (2.7), we obtain

$$G(x) = k(c_1 + c_4 x)^{\frac{c_3}{c_4} + \frac{c_1 c_5}{c_4^2} - \frac{c_1^2 c_6}{c_4^3}} * e^{\frac{1}{2}\frac{x[2c_1c_6 - 2c_4c_5 - c_6c_4x]}{c_4^2}} \qquad \text{when} \quad c_4 \neq 0$$

$$G(x) = ke^{\frac{1}{6}\frac{x[-2c_6x^2 - 3c_5x + 6c_3]}{c_1}} \qquad \text{when} \quad c_4 = 0, c_1 \neq 0$$

$$G(x) = 0 \qquad \text{when} \quad c_1 = c_4 = 0$$

Where k is an arbitrary constant.
Then,

$$\Gamma_1 = [c_1 + c_4 x + 2c_5 t + 4c_6 xt]\frac{\partial}{\partial x} + [2c_4 t + 4c_6 t^2]\frac{\partial}{\partial t} + [c_3 - c_5 x - c_6 x^2 - 2c_6 t]w\frac{\partial}{\partial w}, \quad (2.8)$$

Can be used to solve (2.1) subject to

$$w(x,0) = k(c_1 + c_4 x)^{\frac{c_3}{c_4} + \frac{c_1 c_5}{c_4^2} - \frac{c_1^2 c_6}{c_4^3}} * e^{\frac{1}{2}\frac{x[2c_1c_6 - 2c_4c_5 - c_6c_4x]}{c_4^2}} \quad ; c_4 \neq 0 \quad (2.9)$$

$$\Gamma_2 = [c_1 + 2c_5 t + 4c_6 xt]\frac{\partial}{\partial x} + [4c_6 t^2]\frac{\partial}{\partial t} + [c_3 - c_5 x - c_6 x^2 - 2c_6 t]w\frac{\partial}{\partial w} \quad (2.10)$$

Can be used to solve (2.1) subject to

$$w(x,0) = ke^{\frac{1}{6}\frac{x[-2c_6x^2 - 3c_5x + 6c_3]}{c_1}} \qquad \text{when } c_1 \neq 0 \quad (2.11)$$

And

$$\Gamma_3 = [2c_5 t + 4c_6 xt]\frac{\partial}{\partial x} + [4c_6 t^2]\frac{\partial}{\partial t} + [c_3 - c_5 x - c_6 x^2 - 2c_6 t]w\frac{\partial}{\partial w} \quad (2.12)$$

Can be used to solve (2.1) subject to $w(x,0) = 0.$ (2.13)

3.Examples:

(1) The IVP

$$w_t = w_{xx}; \qquad w(x,0) = x, \quad (2.14)$$

Is of the form (2.1) with IC (initial condition) (2.9) with

$$c_1 = c_2 = c_5 = c_6 = 0, \quad c_3 = c_4 = 1 \text{ and } k = 1.$$

By substituting these constant values in generator (2.8), we obtain

$$\Gamma = [x]\frac{\partial}{\partial x} + [2t]\frac{\partial}{\partial t} + [w]\frac{\partial}{\partial w},$$

Where the infinitesimls are

$$\delta = x, \qquad \tau = 2t \text{ and } \omega = w.$$

With appling the corresponding invariant surface condition (ISC) for equation (2.1)

$$\delta\, w_x + \tau w_t = \omega, \quad (2.15)$$

We obtain

$$x\, w_x + 2t\, w_t = w \quad (2.16)$$

Solving (2.16), we get

$$w(x,t) = x\,\varphi(\zeta), \qquad \text{where} \quad \zeta = \frac{\sqrt{t}}{x} \quad (2.17)$$

Comparison at $t = 0$ with the initial condition (IC) $w(x,0) = x$ give the IC for φ as

$$\varphi(0) = 1.$$

By substituting of (2.17) into (2.14) gives

$$2\zeta^3\varphi'' - \varphi' = 0, \quad (2.18)$$

Subject to $\varphi(0) = 1.$

Solving (2.18) With the aid of Maple, we get

$$\varphi = k_1 + k_2\left(\zeta e^{\frac{-1}{4\zeta^2}} + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}\left(\frac{1}{2\zeta}\right)\right) \quad (2.19)$$

Where k_1 and k_2 are arbitrary constants,
and $\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-\zeta}d\zeta$ is the error function (also called the probability integral)[1].

Consideration of the condition $\varphi(0) = 1$, we obtain $k_1 = 1 - \frac{\sqrt{\pi}}{2}k_2.$

Hence, the solution to the IVP (2.14) is

$$w(x,t) = x\left[\left(1 - \frac{\sqrt{\pi}}{2}k_2\right) + k_2\left(\frac{\sqrt{t}}{x}\,e^{\frac{-x^2}{4t}} + \frac{\sqrt{\pi}}{2}\ \mathrm{ref}\left(\frac{x}{2\sqrt{t}}\right)\right)\right].$$

(2) The IVP

$$w_t = w_{xx}; \qquad w(x,0) = e^x, \tag{2.20}$$

Is of the form (2.1) with IC (initial condition) (2.11) with

$$c_2 = c_4 = c_5 = c_6 = 0, \quad c_1 = c_3 = 1 \text{ and } k = 1.$$

By substituting these constant values in generator (2.10), we obtain

$$\Gamma = \frac{\partial}{\partial x} + w\frac{\partial}{\partial w},$$

Where the infinitesimls are

$$\delta = 1, \qquad \tau = 0 \text{ and } \omega = w.$$

With appling the corresponding invariant surface condition (ISC) for equation (2.1)

$$\delta\, w_x + \tau w_t = \omega, \tag{2.21}$$

We obtain $w_x = w$ (2.22)

Solving (2.22), we get

$$w(x,t) = e^x\, \varphi(\zeta) \quad \text{where} \quad \zeta = t \tag{2.23}$$

Comparison at $t = 0$ with the initial condition (IC) $w(x,0) = e^x$ give the IC for φ as $\varphi(0) = 1$.

By substituting of (2.23) into (2.20) gives $\varphi' - \varphi = 0$ (2.24)

Subject to $\varphi(0) = 1$.

Solving (2.24), we get

$$\varphi = k_1\, e^{\zeta} \quad ; \zeta = t \tag{2.25}$$

Where k_1 is arbitrary constant. Consideration of the condition $\varphi(0) = 1$, we obtain $k_1 = 1$. Hence, the solution to the IVP (2.20) is

$$w(x,t) = e^{x+t}.$$

4. Conclusion

In this paper, we investigate the use of the classical symmetries in solving initial value problems (IVPs). In general, it is believed that in order to be solvable the IVPs, the give condition of the form $w(x,t)|_{t=t_0=0} = G(x)$, need to be left invariant under the the one-parameter of Lie group of transformation that leaves the linear heat equation $w_t = w_{xx}$, Invariant.under this procedure and using the classical symmetry found for governing equation(2.1) , general intial conditions solvable with our partial differential equation(2.1) were established. Some examples are provided to illustrate the method.

References

Abdigapparovich N.O.(2019) Invariant Solutions of Two Dimensional Heat Equation, Journal of Applied Mathematics and Physics,vol.7, 1488-1497.

Al-nassar S.K. (2020) Classical and Nonclassical Symmetry Analysis for The Korteweg-deVries Equation, Journal of Cirtical Reviewes, Vol.7,N(16),PP.2381-2385.

Al-nassar S.K. (2016) Lie Symmetries and Partial Differential Equations (PDEs) ,Journal of Thi-Qar University. Vol.11 No.4,pp.80-89.

Al-nassar S.K. (2015) Lie Symmetries to Ordinary Differential Equations (ODEs) , Journal of College of Education for pure sciences. Vol.5,N.(1), PP.1-18.

Bluman G.W & Kumei S. (1989) , Symmetries and Differential Equations, Springer-Verlag, Berlin_New York.

Basarab-Horwath P. & Zhdanov R. Z. (2001) Initial-value problems for evolutionary partial differential equations and higher-order conditional symmetries, J. Math.Phys. 42, 376–389.

Bluman G.W. & Cole J. D. (1969) The general similarity solution of the heat equation, J. Math. Mech. 18, 1025–1042.

Chandrasekar V.K., Mohanasubha R. & Senthilvelan M.(2015) Symmetries of nonlinear ordinary differential equations: The modified Emden equation as a case study, Pramana - J. Phys.,85(2), pp. 755-787.

Chauhan A., Arora R. & Tomer A.(2020) Lie Symmetries Analysis and Traveling Wave Solutions of Equal Width Wave Equation. Proyecciones, Vol. 39, n.1, pp.179-198.

Cherniha, R. & King, J.R. (2015) Lie and Conditional Symmetries of a Class of Nonlinear (1 + 2)-Dimensional Boundary Value Problems. Symmetry, 7, 1410–1435.

Clarkson, P.A. & Mansfield, E.L.(1993) Symmetry Reductions and Exact Solutions of a Class of Nonlinear Heat Equations. Phys. D, 70, 250–288.

[Clarkson P.A. (1996) Nonclassical symmetry reductions for the Boussinesq equation, Chaos, Sol. Fractals 5, 2261–2301.

Dorodnitsyn, V.A. (1982) Invariant solutions of the nonlinear heat equation with a source. Zh. Vychisl. Mat. Mat. Fiz., 22, 1393–1400.

Englefield M.J.(1992) Boundary condition invariance, in “Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics,” eds. N. H. Ibragimov, M. Torrisi and A. Valenti, pp. 203–208, Kluwer,Dordrecht.

Goard J.M & Al-nassar S.K. (2014) Symmetries for initial value problems, Journal Applied Mathmatics Letters. Vol.28, 56–59.

Goard J.M. (2007) Compatibility and Generalised Conditional Symmetries, Phys. Lett. A, 361(3), 238-241.

Goard J.M (2003) Iterating the classical symmetries method to solve initial value problems, J. Appl. Math. 68, 313–328.

Habibullin I. T. (1996) Symmetry approach in boundary value problems, Nonl. Math. Phys. 3, 147–151.

Hussain E.A. & Alwan Z. M. (2019) Lie Group of Solving System for Partial Differential Equations. Journal of Engineering and Applied Sciences.,14(2), 534.538

Hydon P. E.(2005) “Symmetry analysis of initial-value problems,” Cambridge University Press, Cambridge,.

Hydon P. E. (2000) “Symmetry Methods for Differential Equations”, Cambridge University Press, Cambridge.

Narmanov, O.A. (2018) Lie Algebra of Infinitesimal Generators of the Symmetry Group of the Heat Equation. Journal of Applied Mathematics and Physics, 6, 373-381. https://doi.org/10.4236/jamp.2018.62035

Oleksandr A.P & Roman O.P. (2017) Extended symmetry analysis of generalized Burgers equations, Journal of Mathematical Physics , 58(10),

Olver P.J. (2012) '' Applications of Lie Groups to Differential Equations'', (New-York:Springer),.

Ovsiannikov L. V. (1982) “Group Analysis of Differential Equations,” Academic Press, New York.

Zhang Z. &Chen Y.(2010) Classical and nonclassical symmetries analysis for initial value problems , J.Physics Letters A 374,PP.1117-1120.