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COMPUTATIONAL INVESTIGATION OF A4-GRAPHS FOR CERTAIN LEECH LATTICE GROUPS

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Abstract

In the case of a finite simple group G, and G-conjugacy class of element of order 3, The A4-graph is define as simple graph denoted by A4(G, X) has X as vertex set and $x, y \in X$ are adjacent if and only if x≠y and xy-1 = yx-1.We aim to investigate computationally the structure of theA4(G, X) when$G \cong HS$ or $G \cong McL$ Leech Lattice groups.

Keywords: Leech Lattice Groups; Commuting Graph; Diameter, Cliques.

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1. Introduction

One of the most understandable approaches of examining group structure is to examine a group action on a graph. Similarly, comparable approaches are used to investigate the algebraic characteristics of rings [1]. When examining the algebraic characteristics of finite simple groups, the involution components are crucial. In finite simple groups, however, the members of order 3 are equally essential.For example, a class of elements of order 3 can form the Conway group, which is the group of automorphisms of the Leech lattice in 24 dimensions, and any two of these classes can commute or form the alternating groups A4, A5 or SL2(3),SL2(5). In John Thompson's Quadratic pairs with prime 3, A serious scenario exists in such a class. Several studies in this field have been conducted see ([2,3,4 and 5]). For a finite simple group G and X a G-conjugacy class of G. The A4-graph is stand forA4(G,X) that has X as a vertex set with two vertices $x, y \in X$ is connected if x ≠y, where xy-1 = yx-1. The A4(G,X) is a simple graph that was initially shown by Aubad[6] in that paper anordinary properties of the A4-graph by the side of the structure of A4(G,X) when G≅3D4(2) were scrutinized. One point to consider about A4(G,X) would be that the alternating degree 4 group which is the subgroup $< x,y > \cong A_4$ such that $x,y \in X$ are adjacent. This paper goal to investigate the A4(G,X) $when$ G is either $G \cong HS$ or $G \cong McL$ Leech Lattice groups and X is a G -conjugacy class of elements of order 3.The research involves determining the diameter of the graph as well as investigating the disc structures. Furthermore, the clique and the girth for the graph are computed. Henceforth, we let $G$to be a finite simple and $X = t^G$ where t be an elements of order 3 in G. The conjugation group obviously generates A4-graph automorphisms and is transitive over its vertices. Assume that x be an arbitrary element in$X$ and $i \in \mathbb{Z}+$;(while working with graphs using the usual distance function, This distance function is denoted by d(,)) $\Delta_i(x)$ define as the set of vertices of A4(G,X) with distance equal to i $from$ x. We putG_x(= $C_{G(x)}$) to highlight the centralizer of the element x in G. Evidently, $\Delta_i(x)$sorted into a union of specific $C_{G(x)}$–orbits (this at$G$ acting by conjugation on the set$X$). As a consequence, we will determine the $C_{G(x)}$-orbits of X in order to examine the characteristics of the A4-graph. Lastly, we will extensively depend on Atlas [7] for the G-conjugacy class labels.

2.General Properties

In this part, we will look at some general characteristics of the A4-graph of a finite simple group. We begin with an explanation of the A4-graph, followed by an illustration.

Definition 2.1[6]: For a finite group $G$and an elements of order 3, $t \in G$. Let$X = t^G$be a $G-$ conjugacy class.The A4-graph is a simple graph with the vertex set X, typically indicated by A4 (G,X) with $x, y \in X$ are connected by edge if $x \neq y$and xy-1 = yx-1.

Examples 2.2:

1-Let $G \cong S_6$be a symmetric group of degree 5 and t=(1,2,3), then we have : $X = t^G$ ={(1,2,3),(2,4,3), (1,3,4), (1,4,2), (2,6,3), (1,3,6), (1,6,2), (2,5,3), (1,3,5), (1,5,2) }, Δ2(t)={ (3,4,5), (1,4,5), (2,4,5), (3,6,4), (1,6,4), (2,6,4), (3,5,6), (1,5,6), (2,5,6), (3,4,6), (1,4,6), (2,4,6), (3,5,4), (1,5,4), (2,5,4), (3,6,5), (1,6,5), (2,6,5) , (2,3,4), (1,4,3), (1,2,4), (2,3,6), (1,6,3), (1,2,6), (2,3,5), (1,5,3), (1,2,5), (4,5,6), (4,6,5), (1,3,2)}.

The graph A4 (G,X) is connected with Diam A4 (G,X) =3. The discs structure of the graph A4 (G,X) are: Δ0(t)=t, Δ1(t)={ (2,4,3), (1,3,4), (1,4,2), (2,6,3), (1,3,6), (1,6,2), (2,5,3), (1,3,5), (1,5,2) }, Δ2(t)={(3,4,5), (1,4,5), (2,4,5), (3,6,4), (1,6,4), (2,6,4), (3,5,6), (1,5,6), (2,5,6), (3,4,6), (1,4,6), (2,4,6), (3,5,4), (1,5,4), (2,5,4), (3,6,5), (1,6,5), (2,6,5) , (2,3,4), (1,4,3), (1,2,4), (2,3,6), (1,6,3), (1,2,6), (2,3,5), (1,5,3), (1,2,5) } ,Δ3(t)={ (4,5,6), (4,6,5), (1,3,2)}.This information can be achieved computationally by using the gap package YAGS [10] as we describe in the next procedure which proceeds as follows:

2. Let G≅ D12theDihedral group of order 12. The group G has only one class of element of order 3. Obviously, the graph A4(D6,X) is totally disconnected.

The general properties of the A4-graph can be seen the following results which are proving in [6].

Lemma 2.3: Let G be a finite simple group, and $X = t^G$(such that t has order 3 in G) then A4 (G,X) has the following properties:

1- A4(G,X) is a simple, undirected , regular graph. Moreover, for any two adjacent nodes of A4(G,X) the will generated the group A4.

2- A4($G,X)$ separated into a union of particular $C_{G(x)}$-orbits. Moreover, in case x∈Δ1(t), then tx has the order 3.

Let C be an random G-class, let we look at the set:

XC={x ∈X| tx ∈X}.

We can directly note that when XC≠∅ . This set in this case split into union of certain $C_{G(x)}$ orbits of X. The path of how the non-empty set XC breakis essential to conclude in which $\Delta_i(t)$ the setXCbelong to. By contributing to Class structure constants, measuring the length of the set XC can also be beneficial. The size of the next set are class structure constants:

{(u1, u2) ∈ C1 x C2 | u1u2 = u}

The random elements u is a fixed in the class C3such thatC1, C2, C3 are G-conjugacy classes.The complex character table of G, supplied in the Atlas may now be used to extract these constants, which are now available electronically in modern computer algebra

3.Discs Structure of A4(G,X)

The Leech Lattice Group HS has only one class elements of order 3 named by 3A, while the group McL has two classes of elements of order 3 namely 3A and 3B. For the aforesaid groups the sizes of conjugacy classes for elements of order 3 and the permutation ranks on each class. Also, the structure of the centralizer of t in G which can be seen in [7] are given in the next table. package libraries of Gap[8].

Let we setC1 = C, C2 = X = C3 and u = t, thuswe have

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(g)|}\sum_{i=1}^{s}\frac{\chi_i(p)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(1)}$$

Where in the class C, the element p is fixed. Andχ1, χ2,…, χsrepresent the complex irreducible characters of G.

Table 3.1: Disc Sizes and Permutation Character

Group	Class	Permutation Rank	Size of Class	CG(t)-Structure

HS	3A	399	123200	$C_3 \times S_5$
McL	3A	10	30800	$((C_3\times ((C_3\times C_3): C_3): C_3):SL(2, 5)$
McL	3B	1080	924000	$C_3\times (((C_3\times C_3\times C_3):(C_3 \times C_3\times)): C_3)$

The structures of the Δi(t)of the A4(G,X) are examined in this section for the groups HS and McL.We use a computational technique to get the following results, with help from Gap and the OnLine Atlas[9].

3.1 Disc structure of A4 (McL, 3A)

The Atlas tell us the size of the class $3A$ is 30800 and $C_{G(t)} \cong ((C_3\times ((C_3 \times C_3): C_3): C_3):SL(2,5)$ for G≅McL. The graph A4 (MCL, 3A) is totally disconnected such that there are no edge between any two distance vertices in the graph with 10 $C_{G(t)}$-orbits.

3.2 Disc structure of A4 (HS, 3A)

The Atlas tell us the size of the class 3A is 123200 and $C_{G(t)} \cong C_3 \times S_5$for G≅HS. The graph is connected with 399 $C_{G(t)}$-orbits. G-conjugacy classes of tx for x∈Δi(t); i∈ℕ with the sizes $C_{G(t)}$-orbits forA4(HS, 3A) are given in as we see the following:

3A (15, (18)2, 90, 120)$\in \Delta_1(t)$

2A (15, 90, 120), 2B(18)2, 3A((90)4, (360)2), 4B((90)4, 360), 4C(360)6, 5A (18)2, 5B(60,360),5C(360)27, 6B((90)3,120),7A(360)34, 11AB (360), 12A((90)8, (360)4), 15A(120), 20A(90)2$\in \Delta_2(t)$

1A (1), 3A(20)3, (120)3, (360)2), 4B(90, (360)3), 4C((90)2, 180, (360)8), 5B(120, 180), 5C(360)18, 6B((120)2, (360)14), 7A(360)47, 8A((90)8, (360)20), 8BC(360)14, 10A((90)6, (360)24), 10B(360)2, 11AB(360)27, 12A(360)12, 15A((120)2, (360)8), 20AB((90)2, (360)4) $\in \Delta_3(t)$

A notable result one can see in the above which is the only the classes $1A, 2A, 2B, 5A, 8ABC, 10AB$. The distance between t and x in A4(G,X) decided by the G-conjugacy class to which tx has a place.

3.3 Disc structure of A4 (McL, 3B)

The Atlas tell us the size of the class 3B is 924000 and $C_{G(t)} \cong C_3\times (((C_3\times C_3\times C_3):(C_3 \times C_3\times)):C_3$)for G≅McL. The graph is connected with 1080 $C_{G(t)}$-orbits. G-conjugacy classes of tx for $x \in \Delta_i(t)$; i∈ℕwith the sizes $C_{G(t)}$-orbits forA4($McL, 3B$) are provided in the following:

3B ((81)2, 324)$\in \Delta_1(t)$

2A ((81)2, (324)), 3B((36)6, (972)4), 4A((486)2, (972)10), 5B(972)55, 6B((162)2, 243) ,7AB ((486)6, (972)18), 9AB(324)3, 11AB(972) ∈Δ2(t).

1A(1), 3B((4)2, (12)3, (36)18, (324)6, (486)4), 4A((162)2, (243), (486)2, (972)24), 5B(972)70, 6A((162)2, (324)12, (972)8), 6B((324)12, (486)8, (972)22), 7AB(486, (972)91), 8A ((486)8, (972)96), 9AB((108)8, (324)6, (972)48), 10A((162)2, (486)3, (972)44), 11AB(972)66, 12A((162)2, (486)4, (972)32), 14AB((486)2, (972)16), 15AB((486)2, (972)24), 30AB(162, (972)4∈Δ3(t).

3A ((4)2, (6), 9AB(108)4 ∈Δ4(t).

The classes $\{1A, 2A, 6A, 8A, 10A, 12A, 14AB, 15AB, 30AB\}$ can be used to know the distance between t and x in A4(G, X) such that tx has a place in these classes.

4. $C_{G(t)}$ **−Orbits of A4-graph Description**

Some information about the data in our tables is desired. As said we utilize the class names given within the Atlas book in spite of the fact that we make a few adjustments. we compress the letter portions of the class name when the union of these classes is configured in alphabetical sequence. In Table 4.1, 11AB is abbreviated to $11A \cup 11B$.

1. The first table is about the A4$(HS, 3A)$which contains full information about the $C_{G(t)}$-orbits that are essential to analyze the A4-graph structure.

Table 4.1Orbits Information for A4(HS, 3A)

Class name	Orbits number	Orbits sizes
1A	1	1
2A	3	225
2B	2	36
3A	19	2482
4B	9	1890
4C	17	5400
5A	2	36
5B	4	720
5C	45	16200
6B	20	5670
7A	81	29160
8A	28	7920
8BC	28	5040
10A	30	9180
10B	2	720
11AB	56	10080
12A	24	6480
15A	11	3240
20AB	16	1800

2. The next table involve information about the $C_{G(t)} - orbits$ of A4$(McL, 3A)$as describe below:

Table 4.2Orbits Information for A4 (McL, 3A)

Class name	Orbits number	Orbits sizes
1A	1	1
3A	1	1
3B	1	180
4A	1	1215
5B	1	11664
6A	1	1215
6B	1	0
10A	1	4860
15AB	1	5832

3. The next table involve information about the $C_{G(t)} - orbits$ of A4$(McL, 3B)$ as describe below:

Table 4.3 Orbits Information for A4 (Mcl, 3B)

Class name	Orbits number	Orbits
1A	1	1
2A	3	486
3A	3	14
3B	46	9171
4A	41	35559
5B	125	121500
6A	22	11988
6B	45	29727
7AB	232	109350
8A	104	97200
9AB	138	50868
10A	49	44550
11AB	134	65124
12A	38	33372
14AB	36	16524
15AB	52	24300
30AB	10	4050

5. Girths and Cliques Number

For a finite group G and $X = t^G$ be a class for t the element of order 3 in G. Then girths and the cliques number of the connected A4(G, X) and A4$(G, \Delta_1(t) \cup \{t\})$ are the same. Now we employ this note to build A4$(G, \Delta_1(t) \cup \{t\})$ in gap package YAGS [10]. Thus we obtain the following:

Table 5.1: Girth and clique number

Graph	clique number	Girth
A4(HS, 3A)	16	3
A4 (McL, 3A)	1	∞
A4 (McL, 3B)	16	3

6. Main Results

Theorem 6.1. For G one of the groups of we have the following results:

i. Diam A4$(HS, 3A)$ = 3 and $|\Delta_1(t)|$=261, $|\Delta_2(t)|$=30387, $|\Delta_3(t)|$=92551

ii. A4$(McL, 3A)$ is totally disconnected.

iii. Diam A4 ($McL2, 3B)$ = 4 and $|\Delta_1(t)|$=486, $|\Delta_2(t)|$=114021, $|\Delta_3(t)|$= 808, 614, |Δ4(t)| =878

Proof: Each of Δ_i(t) of A4(G, X) is a union of specific $C_{G(t)} - \text{orbits}$. Thus using the Lemma 2.3 . Then we obtain i, iii. Also we can check computationally to find that $|\Delta_1(t)| =$0 for t element of order 3 in the class $3A$ of the group McL. Thus as the A4-graph is regular we conclude that A4 $(McL, 3A)$ is totally disconnected we obtain the proof of ii.

6.Conclusion:

In this work, the link between two important branches of mathematics, "graph theory" and "group theory," is established. The alternating group A4 as a subgroup for the leech lattice groups $McL \text{ and } HS$ is investigated. For that purpose we use the A4-graph on these groups. Many results have been such that the girth, clique number, diameter and the structure of the graph.

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